Convergence, error estimation and adaptivity in non-elliptic coupled electro-mechanical problems

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Abstract

This paper presents the influence of the lack of ellipticity property on the solution convergence of the coupled electro-mechanical problems. This influence consists in the non-monotonic convergence which can hardly be described analytically. We show that the non-monotonicity depends very much on the energy level of the two component parts of the energy related to the coupled fields of mechanical and electric character. We further investigate the influence of this non-monotonic character of the convergence on the error estimation via equilibrated residual method. We also assess the influence of such convergence on the three- or four-step error-controlled adaptive algorithms. We indicate the methods of practical overcoming the mentioned problems related to the lack of ellipticity and the resulting non-monotonic convergence.

Keywords: ellipticity, coupled problems, solution convergence, error estimation, adaptivity control

1. Introduction

There are two main reasons for undertaking the research of this kind, i.e. to investigate the problems related to non-ellipticity of the operators corresponding to steady-state coupled electro-mechanical problems of piezoelectrics. The first justification results from the more and more frequent contemporary applications of piezoelectric members as transducers, where the mechanical signal, and the corresponding mechanical energy, is converted into electric output and the corresponding electric energy, or vice versa. These two modes of action are performed with the pizoelectric sensors and actuators, respectively. They are associated with the so called direct and inverse piezoelectric phenomena.

The second justification is of scientific character and results from the lack of general convergence theorem for the coupled (electro-mechanical), piezoelectric problems. Such problems are non-elliptic, even in the steady-state case. On the other hand, such theorems are well established in the case of steady-state problems of pure elasticity and pure dielectricity. The latter two problems are elliptic ones. The most significant consequence of the mentioned non-ellipticity is that the coupled problems' approximated solutions belonging to the Sobolev spaces do not guarantee monotonic decrease of the error with the h-refinement or p-enrichment of the approximated solution space.

2. Problem statement

We focus on the electrostatic and elastostatic piezoelectricity being an example of the coupled steady-state phenomena. In this problem we search for $(\boldsymbol{u}, \phi) \in V \times \Psi$, representing coupled fields of displacements and electric potential, such that

$$\begin{cases} B(\boldsymbol{v},\boldsymbol{u}) - C(\boldsymbol{v},\phi) &= L(\boldsymbol{v}) \\ -C(\psi,\boldsymbol{u}) - b(\psi,\phi) &= l(\psi) \end{cases}, \qquad (1)$$

for all $\boldsymbol{v} \in V = \{\boldsymbol{v} \in (H^1(\Omega))^3 : \boldsymbol{v} = \mathbf{0}\}$ and for all $\psi \in \Psi = \{\psi \in H^1(\Omega) : \psi = 0\}$. Above, bilinear forms B, C and b represent the virtual strain, coupling and electrostatic energies, while linear forms L and l denote the virtual works of the external forces and charges, respectively.

In the paper, we would like to assess the possibilities and limitations for the application of the error estimation and adaptivity control procedures, to the case of piezoelectricity introduced above. These algorithms are based on the equilibrated residual methods [1] and three- [2] or four-step [3] adaptive procedures, respectively. They appear to be effective both in the case of elasticity [3] and dielectricity [4]. The author of this work is not aware of any works investigating the electro-mechanical problems of piezoelectricity in the contexts addressed in this paper.

3. Ellipticity

The multi-physics problems are usually related with the conversion of one type of energy into another one. The examples of such problems include piezoelectric media, where the mechanical energy is converted into electric one or vice versa. These media can be described with the partial differential equations of the strong (or local) formulation in such a way that the formulas for the decoupled (or pure) problems, i.e. the elasticity and dielectricity, are coupled via the additional terms appearing in the constitutive relations of the decoupled problems. In this context one may deal with the coupling piezoelectric tensors appearing in the constitutive relations of the elasticity and dielectricity.

In the case of steady-state problems, the decoupled problems of elasticity and dielectricity are described with the elliptic partial differential operators. However, the analogous operator for the coupled problem loses its ellipticity due to different signs of the coupled problems constitutive terms originated from the pure problems of elasticity and dielectricity. This fact reflects different signs of the energy from different sources (mechanical and

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electric ones). In the case of piezoelectricity the symmetric, positive definite and matrices, $\boldsymbol{D}$ and $\boldsymbol{\gamma}$, of elasticity and dielectricity form the symmetric, however non-positive definite matrix of piezoelectricity, i.e.

$$\left[\begin{array}{c} \boldsymbol{\sigma}(\boldsymbol{u}) \\ \boldsymbol{d}(\phi) \end{array} \right] = \left[\begin{array}{cc} \boldsymbol{D} & -\boldsymbol{c} \\ -\boldsymbol{c}^T & -\boldsymbol{\gamma} \end{array} \right] \left[\begin{array}{c} \boldsymbol{\varepsilon}(\boldsymbol{u}) \\ \boldsymbol{E}(\phi) \end{array} \right], \tag{2}$$

where $\boldsymbol{c}$ stands for the coupling electro-mechanical matrix, while $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{d}$ and $\boldsymbol{E}$ represent stress and strain vectors, and the electric displacement and field vectors, respectively.

4. Convergence

Here we check the relations between the energy norm error of the solution to the stationary piezoelectric problems and the number N of degrees of freedom (dofs). The error is defined as a difference of the total potential energies, V and V_r, corresponding to the numerical solution and the reference solution (obtained from the over-killed mesh).

The following problems have been taken into account: the purely elastic problem, the inverse piezoelectric problem (under the dominance of the mechanical part of the potential energy), a mixed piezoelectric problem (with the mechanical and electric parts of the energy of the same order), the direct piezoelectric problem (under the dominance of the electric part of the potential energy), and finally the purely electrostatic case.

The conclusions that can be drawn from our investigation reads as follows. Firstly, the convergence rates are the highest for the purely dielectric case, while the lowest convergence rates correspond to the purely elastic case. In the case corresponding to the electric and mechanical potential energy parts of the same order, the non-monotonic character of the convergence curves can be observed. This can be seen in the piezoelectric plate example (see Fig. 1), where $p =$ var and $m = 1, 2, 4, 8$ is the number of the mesh subdivisions in the longitudinal directions.

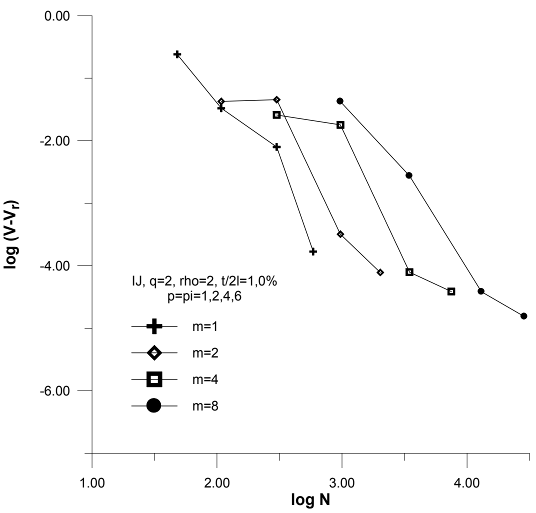

Figure 1: p-Convergence – piezoelectricity case

5. Error estimation

We investigate here the possibility of adoption of the equilibrated residual methods to coupled piezoelectric cases. It was demonstrated in [3] and stated in [4] that the method works effectively in the case of purely elastic and purely dielectric problems, respectively.

As shown in [4] there are no formal obstacles to adopt the structure of the the element residual method to the piezoelectric case. The main problem is however that for the estimated global error measure the upper bounding theorem cannot be proved due to the non-elliptic character of the differential operators used for the obtainment of the solutions of both the local and global piezoelectric problems. In consequence, one can hardly interpret the local and global error measures obtained via the residual approach.

In spite of the theoretical difficulties observed for the coupled cases of piezoelectricity, the numerical tests deliver the following observations. Firstly, the global error indicators are always greater than the true global error values. The effectivity of such indicators is worse than in the pure cases of elasticity and dielectricity. For example, when the effectivity of the pure problems is between 1.0 and 1.5, the effectivity in the analogous piezoelectric problem can be between 3.0 and 4.0 or higher depending on the ratio of the electric and mechanical parts of the potential energy of the global problem.

6. Error-controlled adaptivity

The three- and four-step algorithms of the error-controlled adaptivity for the elliptic problems of pure elasticity and pure dielectricity are based on the convergence theory of the hp-approximation. This theory results from the Cea and FEM interpolation theorems. The theory relates the element error level with the element discretization parameters h and p. As the Cea theorem is valid for elliptic problems only, the adaptivity control for the non-elliptic problem of piezoelectricity with use of this approach cannot guarantee the final mesh error to be close to its admissible target value.

7. Conclusions

Our convergence studies give some hints for the elaboration of the residual-like estimation method and the adaptive scheme controlled by error indicators for the non-elliptic electro-mechanical problems of piezoelectricity.

Effective application of the element residual methods for error estimation to the non-elliptic electro-mechanical systems needs further theoretical and numerical studies.

Taking advantage of the adaptivity control algorithms of the elliptic problems for the coupled piezoelectric problems needs some more insight into this matter through theoretical considerations and numerical tests.

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