# On 2D topology optimization of fatigue constrained problems

**Jacob Oest**[1] **and Erik Lund**[2*]

[1]*Department of Mechanical, Production and Management Engineering, Aalborg University*
*Fibigerstraede 16, DK-9220 Aalborg East, Denmark*
*e-mail: oest@make.aau.dk*

[2]*Department of Mechanical, Production and Management Engineering, Aalborg University*
*Fibigerstraede 16, DK-9220 Aalborg East, Denmark*
*e-mail: el@make.aau.dk*

Abstract

The goal of this work is the development of an efficient gradient-based method for topology optimization of continuum structures with the objective of minimizing mass while taking finite-life high-cycle fatigue constraints into account. The topology optimization uses a density approach and the fatigue constraints are formulated using Palmgren-Miner's linear damage hypothesis, S-N curves, and the Sines fatigue criterion. The large number of local fatigue constraints is reduced by using aggregation functions, and by making use of the accumulative damage rule, an efficient adjoint sensitivity analysis is derived where the amount of adjoint problems to be solved is independent of the amount of cycles in the load spectrum. The approach is limited to linear elastic problems, proportional loading conditions, and quasi-static structural analysis, and the implementation is done for 2D continua. A number of benchmark examples are presented and compared to mass minimized designs with constraints on von Mises stresses.

*Keywords: topology optimization, fatigue constraints, stress constraints*

## 1. Introduction

In this work the goal is to take fatigue constraints into account in density based topology optimization of continuum structures. The inclusion of local constraints in terms of fatigue damage constraints is challenging due to the large number of very nonlinear constraints combined with the high number of design variables. The same computational challenge is faced for stress constrained topology optimization, where the traditional approach is to group the many local constraints into few global stress constraints by use of aggregation functions. The same approach is taken in this work, and by the development of an efficient adjoint sensitivity analysis it is demonstrated that the computational cost associated with fatigue constraints is only slightly increased compared to stress constraints in case of proportional loading conditions.

## 2. Outline of the approach

The approach is limited to linear elastic problems, proportional loading conditions, and quasi-static structural analysis. Thus, the vector of global reference displacements $\hat{\mathbf{u}}$ caused by the reference load vector $\hat{\mathbf{P}}$ is found by solving the equilibrium state equation:

$$\mathbf{K}\left(\tilde{\mathbf{x}}\left(\mathbf{x}\right)\right)\hat{\mathbf{u}} = \hat{\mathbf{P}} \tag{1}$$

where $\mathbf{x}$ contains density variables $x_e$ for each element $e$ and $\tilde{\mathbf{x}}$ are filtered density variables obtained using standard density filtering. $\mathbf{K}$ is the interpolated global stiffness matrix computed using a modified SIMP approach where a penalization factor $p = 3$ is applied when interpolating the modulus of elasticity $E_e$ by:

$$E_e\left(\tilde{x}_e\left(\mathbf{x}\right)\right) = E_{min} + \tilde{x}_e(\mathbf{x})^p\left(E_0 - E_{min}\right),\ \mathbf{x} \in [0;1] \tag{2}$$

$E_0$ is the Young's modulus of the isotropic material and $E_{min} \ll E_0$ is a lower bound on the modulus, representing the material stiffness of a void region. The element reference stress $\hat{\boldsymbol{\sigma}}_e$ caused by the reference load can be found by:

$$\hat{\boldsymbol{\sigma}}_e = \tilde{x}_e(\mathbf{x})^q\mathbf{E}\mathbf{B}\hat{\mathbf{u}}_e \tag{3}$$

$\mathbf{E}$ is the constitutive matrix for full material density, $\mathbf{B}$ is the strain-displacement matrix, and the exponent $q < 1$ is introduced to address the singularity phenomena by relaxing the design space [1, 2].

Based on the reference solution obtained for each reference load, linear scaling is applied for the time-varying quasi-static proportional loading condition, and the amplitude and mean displacement and stress values are determined from traditional rainflow-counting. The stress-based Sines criterion is applied as a multi-axial fatigue criterion in order to obtain an equivalent uniaxial stress $\tilde{\sigma}_{e_i}$ for a given stress cycle $i$. This equivalent stress can be related to an estimated amount of cycles to failure $N_{ei}$ using Basquin's equation that represents a log-log straight line $S - N$ relationship. Expressed in stress reversals, the $S - N$ curve is given by:

$$\tilde{\sigma}_{e_i} = \sigma_f^{'}(2N_{ei})^b,\ \ \forall e, i \tag{4}$$

Here $\sigma_f^{'}$ is the fatigue strength coefficient and $b$ is the fatigue strength exponent, corresponding to the slope of the log-log $S - N$ curve. This material-specific equation adds a very large non-linearity to the analysis. In order to accumulate the damage for the entire load spectrum, Palmgren-Miner's linear damage hypothesis is applied, such that the accumulated fatigue damage $D_e$ in element $e$ is found by collecting all the fractions of damage $D_{e_i}$ caused by each load cycle $i$:

$$D_e = \sum_{i=1}^{n_{RF}} D_{e_i} = c_D\sum_{i=1}^{n_{RF}}\frac{n_i}{N_{ei}} \leq \eta,\ \ \forall e \tag{5}$$

Here $n_i$ is the amount of reversals for all stress cycles $i = 1, \ldots, n_{RF}$, and $c_D$ is a scaling parameter making the load history representative of the entire lifetime. The upper limit $\eta$ is set

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to 1, and a global fatigue constraint is introduced by combining all the element fatigue damages using a P-norm function.

The analytical sensitivity analysis is derived using the adjoint approach, and the need for solving the adjoint equation for each stress cycle $i$ is eliminated by solving for the sum of all scaled Lagrange multipliers. This makes it possible to solve for very large load series without a significant increase in computational cost, and the approach is easily implemented for other stress-based fatigue criteria.

## 3. Numerical Example

A 2D double-L shaped membrane structure is considered in order to demonstrate the difference between stress constrained and fatigue constrained minimum mass designs, see definition of problem in Figure 1 a). The plate has a thickness of $t$ = 0.02 m, and the material applied is AISI 1020 HR steel. The material values are taken from [3]. The static reference load applied for the stress constrained example is $\hat{P} = 75\ kN$, distributed onto six elements at the vertical edge. In the fatigue constrained optimization the load spectrum contains $k = 1,000$ loads, and is defined by $P_k = rand(\hat{P}, -\hat{P}/2), \forall k$. The default random number generator in MATLAB is applied, such that the loading conditions can be recreated. The damage is scaled with $c_D = 10,000$. The design domain is discretized using a discretization of $9,464$ 4-node bilinear elements and the aggregation function is assigned a high $P$-norm value of $P = 12$.

The von Mises constrained design yields, as expected, a symmetric design that minimizes the stress concentrations at the edges, see Figure 1 b)-c). The fatigue constrained design is asymmetric, and there is no connection to the upper fixed boundary, see Figure 1 e)-f). This is due to the difference in mean stress effects caused by the loading condition, which is not fully reversed. For comparison, the von Mises stress distribution is also shown for the fatigue constrained problem in Figure 1 f). The distribution of element fatigue damages is seen to be very uniform, thereby demonstrating the success of the optimization, see Figure 1 g). It is seen by the iteration histories in Figure 1 d) and h) that several more design iterations are needed for the fatigue constrained problem which is due to the increased nonlinearity of the problem.

## 4. Concluding Remarks

A general method for finite-life fatigue optimization in case of proportional loading conditions that includes the entire high-cycle fatigue analysis directly in the optimization has been presented. Only one adjoint vector per constraint equation has to be solved for every reference load vector. The method is not limited to 2D applications, and it is generic in the sense that other stress-based fatigue criteria can be applied.

## References


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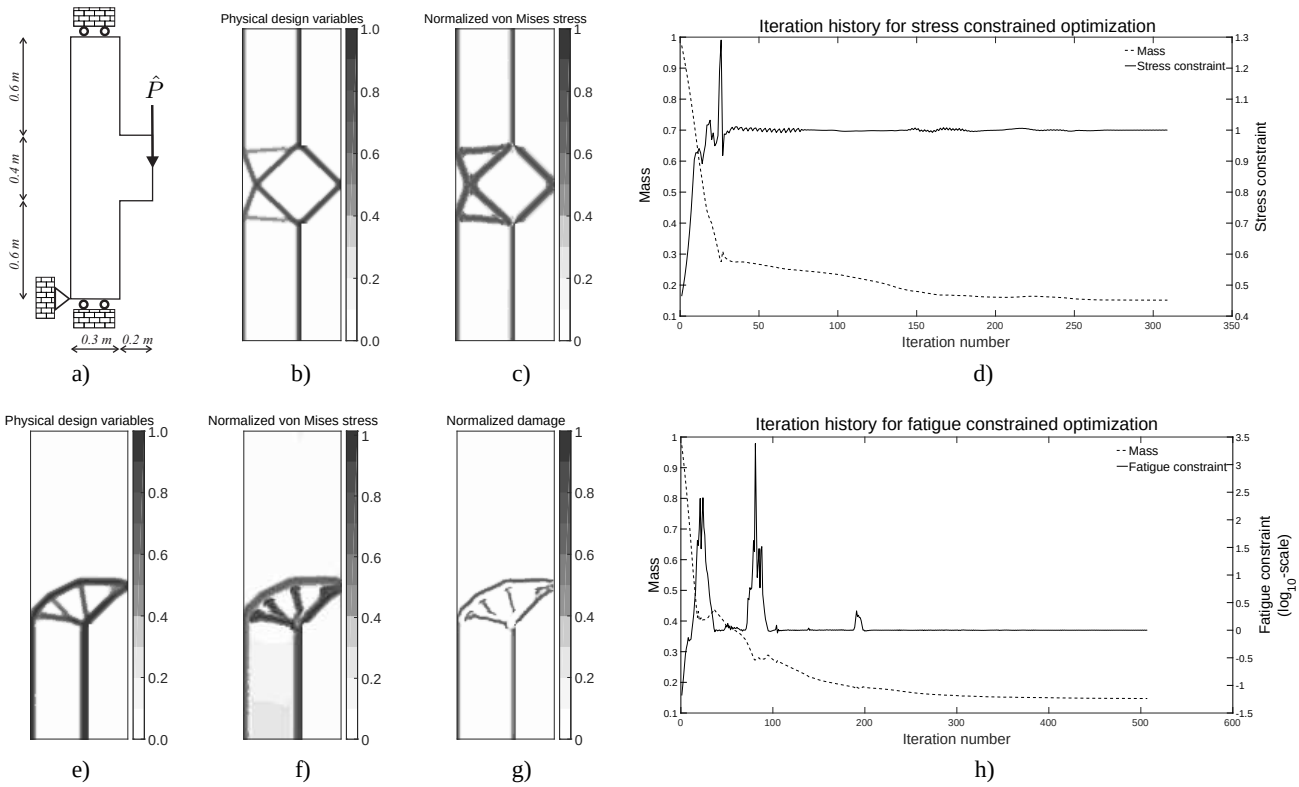


Figure 1: Double-L membrane problem. The results from the stress constrained optimization are given in b), c) and d), while results from the fatigued constrained problem are given in e), f), g) and h). In both cases the mass is reduced to 15% of the full density design.