Stability calculation of an elastic rod with delayed boundary conditions

Li Zhang[1] **and Gabor Stepan**[2]*****

[1] *College of Aerospace Engineering, Nanjing University of Aeronautics and Astronautics*
Yudao 29, 210016 Nanjing, China
e-mail: zhangli@nuaa.edu.cn

[2] *Department of Applied Mechanics, Budapest University of Technology and Economics*
Műegyetem 5, 1521 Budapest, Hungary
e-mail:stepan@mm.bme.hu

Abstract

The stability of an elastic rod subjected to delayed feedback is analysed. The governing equation of this system is the 1D wave equation with delayed boundary conditions. By means of D'Alembert solution, the exact stability chart is presented in the parameter plane of the feedback gain and time delay for both undamped and damped cases. The closed form analytical result explains the extreme sensitivity of the dynamics with respect to time delay, which also shows the numerical difficulty of finite degree of freedom approximations for the continuum systems.

Keywords: wave equation, delayed feedback, delayed boundary condition, stability chart

1. Introduction

When dealing with dynamic problems of continua, such methods as modal decomposition and finite element methods are usually used, which transform the original partial differential equation into a set of ordinary differential equations. However, as the number of degrees of freedom (DoF) increases, difficulties may arise whether the finite DoF approximation tends to the continuum model. This is an important issue often addressed in the literature [1, 2].

In practical applications, time delays are almost inevitable if a system involves any feedback loops. The presence of the unavoidable time delay in the controlled continuum systems may lead to governing equations that consist of partial differential equations with delayed boundary conditions [3]. As a representative example, a basic microphone-amplifier-loudspeaker system was modelled as a linear elastic rod with time delay in the control loop, which resulted in 1D wave equation with delayed boundary conditions [4]. In their study, the finite DoF approximations did not show convergence in the stability charts for certain small stable domains that were found numerically in the delayed continuum model.

The aim of this paper is to conduct exact stability analysis for the delayed boundary value problem modelling a controlled elastic rod. It also provides an explanation why these kinds of stability charts cannot be obtained by finite DoF approximations of the continuum rod.

2. Mechanical model

A linearly elastic prismatic rod is subjected to a longitudinal load at its left end, (see Fig. 1). The rod has length *l*, cross sectional area *A*. Its material has density ρ and modulus of elasticity *E*. *K* is the dimensionless feedback gain and τ is the the time delay in the feedback loop. The governing equation of the longitudinal vibrations of the rod takes the following form:

$$\ddot{u}(x,t) - c^2 u''(x,t) = 0 \tag{1}$$

where c is the speed of sound given by $c = \sqrt{E/\rho}$. The two boundary conditions are:

$$\begin{aligned} &u(l,t) = 0 \\ &u'(0,t) - Ku'(l,t-\tau) = 0 \end{aligned} \tag{2}$$

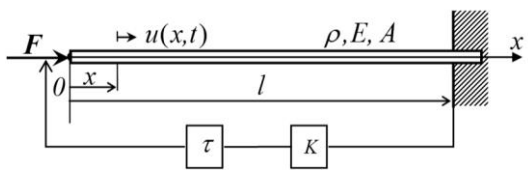

Figure 1: Mechanical model of an elastic rod subjected to delayed feedback.

With the D'Alembert solution in the form of

$$u(x,t) = f(t - x/c) + g(t + x/c) , \tag{3}$$

Eqn (3) is transformed to a neutral delay differential equation with respect to the unknown function *f* as follows:

$$\dot{f}(t) + \dot{f}(t-2T) - 2K\dot{f}(t-T-\tau) = 0 , \tag{4}$$

where $T = l/c$, is the time needed for a wave of speed *c* traveling along the rod.

3. Stability analysis

The exponential stability of the equilibrium is represented in the parameter plane of the delay ratio τ/T and the feedback gain *K*. Based on the existing instability results for rationally independent delays [5], and the results of analyses of all the possible combinations of commensurable delays, it is shown that only the following zero-measure parameter sets can guarantee the exponential asymptotical stability of the

*The study was supported by National Natural Science Foundation of China under Grants No.11302098, the Hungarian-Chinese Bilateral Scientific and Technological Cooperation Fund under grant No. TET 12 CN-1-2012-0012, Research Fund of the State Key Laboratory of Mechanics and Control of Mechanical Structures (Nanjing University of Aeronautics and Astronautics) Grant No. MCMS-0116K01 and China Scholarship Fund.

equilibrium of the wave equation (1) with boundary conditions (2,3),

$$\tau / T = 4n-3, \quad 0 < K < \cos\left(\frac{2n-2}{4n-3}\pi\right), \quad (n = 1, 2, \cdots), \tag{5}$$

and

$$\tau / T = 4n-1, \quad \cos\left(\frac{2n}{4n-1}\pi\right) < K < 0, \quad (n = 1, 2, \cdots), \tag{6}$$

According to the exact stability results as shown in Eqn (5) and Eqn (6), we obtain the exact stability chart of the wave equation with delayed boundary conditions, (see Fig.2).

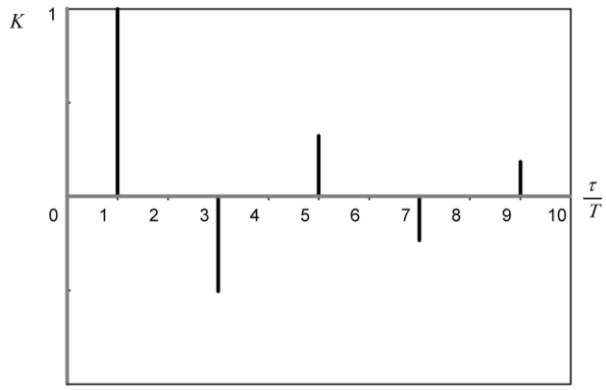

Figure 2: Stability chart in the plane of the delay components ratio τ / T and the feedback gain *K*.

In Fig. 2, white region refers to exponentially unstable domains. Thick black lines show exponentially asymptotically stable domains, and grey thick line represents marginally stable domain. The stability chart indicate that any small variation of the delay may cause the sudden change of the system from exponentially asymptotically stable to exponentially unstable, which follows the discontinuity property of the spectrum with respect to the delays [6].

4. Stability analysis of the damped system

Consider the case of viscous internal damping with a factor $\eta > 0$ that describes the damping forces proportional to the internal elastic forces. Then the governing equation of the damped elastic rod has the following form:

$$\ddot{u}(x,t) - \eta c^2 \dot{u}''(x,t) - c^2 u''(x,t) = 0 \tag{5}$$

It has the same delayed boundary conditions (2,3) as in the undamped case.

The stability analysis provides the equations for the critical parameters τ and *K*:

$$K_{\mathrm{cr}}^2 = \sinh^2\left(\frac{\omega T}{\sqrt{2}} \frac{\sqrt{\sqrt{\eta^2\omega^2+1}-1}}{\sqrt{\eta^2\omega^2+1}}\right) + \cos^2\left(\frac{\omega T}{\sqrt{2}} \frac{\sqrt{\sqrt{\eta^2\omega^2+1}+1}}{\sqrt{\eta^2\omega^2+1}}\right), \tag{6}$$

$$\tan(\omega\tau_{\mathrm{cr}}) = -\tanh\left(\frac{\omega T}{\sqrt{2}} \frac{\sqrt{\sqrt{\eta^2\omega^2+1}-1}}{\sqrt{\eta^2\omega^2+1}}\right) \tan\left(\frac{\omega T}{\sqrt{2}} \frac{\sqrt{\sqrt{\eta^2\omega^2+1}+1}}{\sqrt{\eta^2\omega^2+1}}\right) \tag{7}$$

The stability chart is presented in Fig. 3 for different dimensionless damping factor 0, 0.001, 0.01. The stability region widens around the stability lines of the undamped system as the internal damping increases.

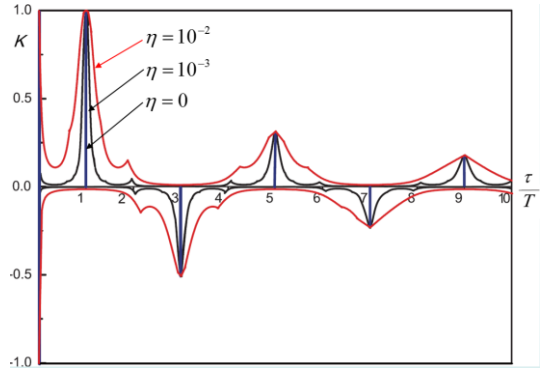

Figure 3: Stability chart for dimensionless damping factor $\eta = 0$, $\eta = 0.001$, $\eta = 0.01$.

5. Conclusion

The exact stability chart is presented in the plane of the feedback gain and the delay. It consists of discrete, uniformly spaced zero-measure lines at odd delay ratios where the system is exponentially stable. The slightest perturbation of the delay parameters leads to exponentially unstable systems. In the presence of internal damping, the stable domains open up to finite regions of exponential stability.

In spite of the large computational efforts based on finite DoF approximations like finite element analysis and experimental modal analysis, the dynamical properties of these delayed boundary systems are still unexplored. The closed form analytical results also explain why the finite DoF approximations do not converge.

References

[1] Holmes, P., Domokos, G. and Schmitt J., Szeberenyi I., Constrained Euler buckling: an interplay of computation and analysis, *Comput. Methods*, 170, pp. 175-207, 1999.

[2] Ito, K. and Turi, J., Numerical methods for a class of singular integro-differential equations based on semigroup approximation, *SIAM J. Numer. Anal.*, 28, pp. 1698-1722, 1991.

[3] Haraguchi, M. and Hu H.Y., Stability analysis of a noise control system in a duct by using delay differential equation, *Acta Mech. Sinica*, 25, pp. 131-137, 2009.

[4] Kidd, M. and Stepan, G., Delayed control of an elastic beam, *Int. J. Dyn. Control*, 2, pp. 68-67, 2014.

[5] Hale, J.K. and Verduyn Lunel, A.M., Strong stabilization of neutral functional differential equations, *IMA J. Math. Control Inf.*, 19, pp. 5-23, 2002.

[6] Datko, R., Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks, *SIAM J. Control Optimiz.*, 26, pp. 697-713, 1988.