Material orientation design of planar structures with prescribed anisotropy classes. Study of rhombic systems

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Abstract

The paper deals with the minimum compliance problem of 2D structures made of a non-homogeneous elastic material. An optimum design is found for a selected design domain and a single load case. In the first part of the paper comparison between solutions of Free Material Design(FMD), Cubic Material Design(CMD) and Isotropic Material Design(IMD) is shown for simply supported plane structure subjected to a concentrated force. The isoperimetric condition fixes the value of the cost of the design expressed as the integral of the trace of the Hooke tensor. In the second part of the paper the material design approaches are extended to rhombic system in 2D. For the rhombic system the material properties of the structures are set, the design variables being the trajectories of anisotropy directions which in 2D are described by one parameter. In Orthotropic Orientation Design (OOD) no isoperametric condition is used.

Keywords: topology optimization, cubic material design, orthotropic orientation design

1. Introduction

The aim of the material design is to construct - within a given feasible domain - the stiffest structure capable of transmitting a given load to a given boundary support by appropriate choice of the material characteristics. FMD in its original formulation involves no restrictions on the components of the Hooke tensor, apart from necessary symmetries and positive semi-definiteness conditions. The isoperimetric condition fixes the value of the cost of the design expressed as the integral of the trace of the Hooke tensor. A natural extension of FMD is a priori imposing certain material symmetries. The strongest assumption is isotropy-this modification has been proposed by Czarnecki and Wawruch [1] and it is called the Isotropic Material Design (IMD). In this formulation only design variables are the bulk k and shear μ moduli in each point of the feasible domain. However, the real bodies are usually not isotropic. A natural extension of the IMD method towards designing of new materials or describing available materials is an approach proposed by Czubacki and Lewiński[2] in which the material is viewed as endowed with a microstructure of cubic symmetry. The total compliance of the structure is expressed by the Castigliano formula:

$$Y = \min_{\boldsymbol{\tau}\in\Sigma(\Omega)} \int_{\Omega} \boldsymbol{\tau}\cdot(\boldsymbol{C}^{-1}\boldsymbol{\tau})\,\mathrm{d}x \tag{1}$$

where $\Sigma(\Omega)$ is a set of statically admissible stress fields and the form of the Hooke tensor $\boldsymbol{C}$ differs for anisotropy(FMD), cubic symmetry(CMD) and isotropy(IMD).

Due to an exceptional mathematical structure of the minimum compliance problems(FMD,CMD,IMD) it is possible to eliminate all design variables, which leads to an auxiliary problem of the form:

$$Z = \min\left\{ \int_{\Omega} |||\boldsymbol{\tau}|||\mathrm{d}x \;\middle|\; \boldsymbol{\tau}\in\Sigma(\Omega) \right\} \tag{2}$$

The latter problem is expressed by the load, kinematic boundary conditions and the design domain.

The integral in (2) for IMD and CMD reads:

$$|||\boldsymbol{\tau}||| = \alpha|\mathrm{tr}\boldsymbol{\tau}| + \beta\|\mathrm{dev}\boldsymbol{\tau}\| \tag{3}$$

where $\mathrm{tr}\boldsymbol{\tau}$ is the trace of $\boldsymbol{\tau}$, $\mathrm{dev}\boldsymbol{\tau}$ is the deviator of the stress and α, β are respectively equal to: $\sqrt{2}/2$, $\sqrt{2}$ for IMD, and $\sqrt{2}/2$, 1 for CMD. Due to the similarity of CMD and IMD methods a stress field which minimizes (2) can be found by using the numerical algorithm described in Czarnecki and Wawruch [1].

The minimal compliance is expressed with the formula: Z^2/Λ in which Z is a number expressed by the minimization problem (2) and Λ is the integral of the trace of the Hooke tensor.

2. Case study: a simply supported deep beam under a single force load

Numerical computation of simply supported structure shown in Fig.1a is performed to compare optimal compliances obtained by three methods: FMD, CMD and IMD.

Table 1: The comparison of the optimal relative compliances

Mesh	FMD	CMD	IMD
80x20	12.01	22.39	34.74
40x10	12.34	22.95	34.90

In Fig.1b,c the optimal layouts of moduli a and c cf.[2] of the cubic material are presented. In Table 1 total relative compliance of the structure for different meshes and different methods are compared. It can be seen that the smallest compliance is obtained by FMD, this is because FMD has no restrictions on the components of the Hooke tensor, apart from necessary symmetries and positive semi-definiteness conditions, which leads to better adaptaion of FMD structure to transmit the load to the boundary than other methods. On the other hand the compliance obtained

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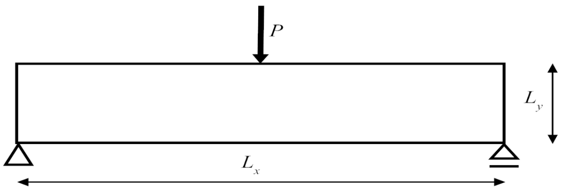
by IMD - for which the strongest assumption is imposed on the Hooke tensor - is the highest.



(a) Geometry of the problem $L_x = 0.4L$, $L_y = 0.1L$.

(b) elastic modulus a/E_0.

(c) elastic modulus c/E_0.

Figure 1: Geometry and the optimal layouts of moduli obtained by CMD; $E_0 = \Lambda/|\Omega|$.

3. Rhombic system or orthotropy

The aim of this part of the paper is to put forward the Orthotropic Orientation Design(OOD) in which material of the structure has orthotropic properties. It this work 2D case of OOD is delivered. The main difference between the material design problem and OOD is that in OOD elastic material moduli a, b, c, l are set a priori, which implies that in this formulation no isoperimetric condition is used. At each point of the structure the orientation of the material defined by a pair of mutually orthogonal unit vectors ($\boldsymbol{n}$,$\boldsymbol{m}$) are to be determined. The Hooke tensor of a material of rhombic symmetry is represented by the celebrated formula by Walpole[4]:

$$\boldsymbol{A} = a\boldsymbol{U}_{11} + b\boldsymbol{U}_{22} + c(\boldsymbol{U}_{12} + \boldsymbol{U}_{21}) + l\boldsymbol{V} \tag{4}$$

where $a > 0$, $b > 0$, $ab - c^2 > 0$, $l > 0$ and fourth-rank tensors $\boldsymbol{U}_{11}, \boldsymbol{U}_{22}, \boldsymbol{U}_{12}, \boldsymbol{U}_{21}, \boldsymbol{V}$ are defined as follows:

$$\begin{aligned} &(\boldsymbol{U}_{11})_{ijkl} = n_i n_j n_k n_l, \qquad (\boldsymbol{U}_{22})_{ijkl} = m_i m_j m_k m_l \\ &(\boldsymbol{U}_{12})_{ijkl} = (\boldsymbol{U}_{21})_{ijkl} = n_i n_j m_k m_l \\ &\boldsymbol{V}_{ijkl} = \frac{1}{2}(m_i n_j + n_i m_j)(m_k n_l + n_k m_l) \end{aligned} \tag{5}$$

The spectral representation of the inverse of $\boldsymbol{A}$ reads:

$$\boldsymbol{A}^{-1} = a_1\boldsymbol{U}_{11} + b_1\boldsymbol{U}_{22} + 2c_1\boldsymbol{U}_{12} + \frac{1}{l}\boldsymbol{V} \tag{6}$$

where $a_1 = b/d$, $b_1 = a/d$, $c_1 = -c/d$ and $d = ab - c^2$.

The minimum compliance problem of the OOD is formulated as:
for given elastic moduli a, b, c, l find the orthogonal trajectories of the vector fields ($\boldsymbol{m}$, $\boldsymbol{n}$) at each point of the feasible domain Ω, such that the structure is characterized by the smallest total compliance among all structures designed in the same feasible domain and capable of transmitting the same load to the same boundary.

By replacing tensor $\boldsymbol{C}^{-1}$ by $\boldsymbol{A}^{-1}$ in (1) we express the total compliance of the structure made of orthotropic material. Upon the change of $\boldsymbol{\tau}$ into $\boldsymbol{\sigma}$ the integrand in (1) can be written as:

$$\boldsymbol{\sigma}.\boldsymbol{A}^{-1}\boldsymbol{\sigma} = a_1(\sigma_n)^2 + b_1(\sigma_m)^2 + 2c_1\sigma_n\sigma_m + \frac{2}{l}\sigma_{\overline{mn}} \tag{7}$$

where $\sigma_n = \sigma_{ij}n_i n_j$, $\sigma_m = \sigma_{ij}m_i m_j$, $\sigma_{\overline{mn}} = \sigma_{ij}n_i m_j$.

Let σ_1, σ_2 are principal stresses, $\overset{1}{\boldsymbol{e}}$, $\overset{2}{\boldsymbol{e}}$ being the unit eigenvectors. Thus,

$$\begin{aligned} \sigma_n &= (\sigma_1 \overset{1}{e}_i \overset{1}{e}_j + \sigma_2 \overset{2}{e}_i \overset{2}{e}_j) n_i n_j = \sigma_1 x^2 + \sigma_2 y^2 \\ \sigma_m &= \sigma_1 y^2 + \sigma_2 x^2, \quad \sigma_{\overline{mn}} = xy(\sigma_2 - \sigma_1) \end{aligned} \tag{8}$$

where $x = (\boldsymbol{n}\cdot \overset{1}{\boldsymbol{e}}) = (\boldsymbol{m}\cdot \overset{2}{\boldsymbol{e}})$ and $y = (\boldsymbol{n}\cdot \overset{2}{\boldsymbol{e}}) = -(\boldsymbol{m}\cdot \overset{1}{\boldsymbol{e}})$

By substituting (8) into (7) one gets:

$$\begin{aligned} &W(\boldsymbol{\sigma}, x, y) = \boldsymbol{\sigma}.\boldsymbol{A}^{-1}\boldsymbol{\sigma} = a_1(\sigma_1 x^2 + \sigma_2 y^2)^2 + \\ &b_1(\sigma_1 y^2 + \sigma_2 x^2)^2 + 2c_1(\sigma_1 x^2 + \sigma_2 y^2)(\sigma_1 y^2 + \sigma_2 x^2) + \\ &\frac{2}{l}x^2 y^2 (\sigma_2 - \sigma_1)^2 \end{aligned} \tag{9}$$

Since $x^2 + y^2 = 1$ only one parameter $x \in [0, 1]$ may be used to calculate the optimal orientation of the material. It can be easily proven that minimum of W is obtained either for $x = 0$ or $x = 1$, which means that optimal orientation of the material coincides with the directions of principal stresses.

Thus, for the Orthotropic Orientation Design (2) one obtains:

$$Z = \min_{\boldsymbol{\tau} \in \Sigma(\Omega)} \int_\Omega F(\boldsymbol{\tau})\, \mathrm{d}x \tag{10}$$

For the case of $a_1 > b_1$ the integrand $F(\boldsymbol{\tau})$ equals:

$$F(\boldsymbol{\tau}) = \begin{cases} W(\boldsymbol{\tau}; 1, 0) & \text{if } \tau_1 > \tau_2 \\ W(\boldsymbol{\tau}; 0, 1) & \text{otherwise} \end{cases} \tag{11}$$

For the case of $a_1 < b_1$ the integrand in (10) should be changed appropriately.

4. Conclusions

The method described in the paper may be used for 3D printing, see the paper by Zegard and Paulino [5] in which alternative topology optimization methods have been adopted.

The problem (10), (11), which is the key towards OOD, has similar mathematical structure to the compliance optimization problem concerning the materials with constitutive laws being dissymmetric with respect to tension-compression [3].

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