Material point analysis of three-dimensional silo flow problem

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Abstract

Three-dimensional problem of granular flow in a silo has been considered. To solve such a large strain problem, the material point method has been applied. An example of granular flow in a silo of quadrilateral cross-section with a convergent hopper has been shown. The numerical results have been compared with the experimental ones; good agreement between both the outcomes has been obtained.

Keywords: granular flow, material point method, arbitrary Lagrangian–Eulerian formulation

1. Introduction

Gravitational flow of a granular material in a silo has been considered. Finding the flow pattern and interactions between the flowing material and silo walls and evaluation of flow rate are interesting from the engineering view point. In the analysis, the granular material has been treated as a solid body—an elastic–viscoplastic, non-associative material model with the Drucker–Prager yield condition has been utilised. The dynamic, large strain problem has been solved by the use of the material point method (MPM). The method was applied successfully to large strain problems of solid mechanics, e.g. [2, 4]. The standard finite element method (FEM) formulated in the Lagrangian description of motion is not a sufficiently robust tool for analysis of the full process of silo discharge due to large mesh distortions which deteriorate the accuracy of the approximate solution. A mesh re-zoning technique aided to the Lagrangian approach is not a comprehensive remedy to overcome the problem as it need mapping state variables from the distorted mesh to a newly generated one which is an additional source of computational inaccuracies. In the material point method, state variables are traced at a set of points (called material points) representing subregions of the analyzed body on which the region initially representing the body is divided. The state variables are calculated by the use of a finite element mesh (called a computational mesh) which can be defined in a arbitrary way which means that mesh distortions are avoided. Due to the fact that two kinds of spatial discretisation are used—material points and a computational mesh—the method reveals features of an arbitrary Lagrangian–Eulerian description of motion which makes the method very flexible in applications. Such problems like self-contact of the granular material and flow around obstacles (inserts, e.g. [4]) are much easier to solve than in the case of other methods.

2. Problem description

The problem of silo flow is formulated as a dynamic one. The initial displacement and stress fields caused by the gravity forces are found by the solution of the corresponding quasi-static problem.

The elastic–viscoplastic constitutive relations have been used to model the mechanical behaviour of the granular material. The Drucker–Prager yield condition and a non-associative flow rule implying the plastic incompressibility of the material are applied in the constitutive model. Let f denote the yield function $f(\sigma_{ij}) = q - mp - k$, where $m = 18\sin\varphi/(9-\sin^2\varphi)$ and $k = 18\,c\cos\varphi/(9-\sin^2\varphi)$ are functions of the angle of internal friction, φ, and cohesion, c, p and q are invariants of the stress tensor, $p = -\frac{1}{3}\sigma_{ii}$, $q = \sqrt{\frac{3}{2}s_{ij}s_{ij}}$, where $s_{ij} = \sigma_{ij} + p\,\delta_{ij}$ denotes the deviatoric part of the stress tensor. The constitutive relations for the elastic–viscoplastic model are as follows:

$$\dot{p} = K\,d_{kk}, \qquad e^{\mathrm{e}}_{ij} = \frac{1}{2\,G}\,\overset{\triangledown}{s}_{ij}, \qquad e^{\mathrm{vp}}_{ij} = \gamma\,\langle\Phi(f)\rangle\,\frac{\partial g}{\partial s_{ij}} \tag{1}$$

where $g \equiv q$ is the plastic potential. The following notation is used above: $d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$ is the rate-of-deformation tensor, e^{e}_{ij} and e^{vp}_{ij} are parts of its deviator, $e_{ij} = d_{ij} - \frac{1}{3}\,d_{kk}\,\delta_{ij}$, the elastic and plastic ones, respectively, $\overset{\triangledown}{\sigma}_{ij} = \dot{\sigma}_{ij} - \sigma_{ik}\,\omega_{kj} - \sigma_{jk}\,\omega_{ki}$ is the Zaremba–Jaumann rate of the stress tensor, $\omega_{ij} = \frac{1}{2}(v_{j,i} - v_{i,j})$ the spin, K and G are the bulk and shear moduli, respectively. Symbol γ in Eq. (1) denotes the viscosity parameter while the function defining the law of plastic flow has the following form:

$$\Phi\left(f(\sigma_{ij})\right) = \left(\frac{q - m\,p - k}{m\,p + k}\right)^{N}, \quad N > 0,$$

$\langle\Phi(f)\rangle = \Phi(f)$ if $f > 0$ and $\langle\Phi(f)\rangle = 0$ othewise.

3. Material point solution

The considered problem is solved by the use of the material point method, where—as in the standard finite element method—the principle of virtual work is the starting point for the formulation of the method. The equation of virtual work has the following form:

$$\int_\Omega (\varrho\,a_i\,w_i + \sigma_{ij}w_{i,j})\,\mathrm{d}x = \int_\Omega \varrho\,b_i\,w_i\,\mathrm{d}x + \int_{\Gamma_\sigma} t_i\,w_i\,\mathrm{d}s + \int_{\Gamma_c} \sigma_{ij}\,n_j\,w_i\,\mathrm{d}s \quad \forall \boldsymbol{w} \in V_0 \tag{2}$$

where V_0 denotes the space of kinematically admissible fields of displacements, Γ_σ the part of boundary Ω where tractions are given, and Γ_c the part of the boundary where the frictional contact problem is to be solved; the friction phenomenon is described by the Coulomb model. Let us introduce a division of the region occupied initially by the analysed body into a set of subregions—each of them represented by one of its points called a material point. We assume that the mass density field is expressed as follows: $\varrho(\boldsymbol{x}) = \sum_{P=1}^{N} M_P\, \delta(\boldsymbol{x} - \boldsymbol{X}_P)$, where M_P and $\boldsymbol{X}_P$ denote the mass and the position of the Pth material point, $\delta(\boldsymbol{x})$ is the Dirac δ-function. Besides this space discretisation (of Lagrangian type), another one—an Eulerian finite element mesh (called a computational mesh) covering the virtual position of the analysed body—is also used. This mesh can be changed arbitrarily during calculations or remain constant. After substituting mass density representation to the equation of virtual work (2) and expressing the field of acceleration, a_i, and the weight functions, w_i, by the shape functions and nodal parameters, defined on the computational mesh, as in the finite element method, we obtain the following system of dynamic equations:

$$\mathbf{M}\,\mathbf{a} = \mathbf{F} + \mathbf{F}_c - \mathbf{R}, \tag{3}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{a}$ the vector of nodal accelerations, $\mathbf{F}$, $\mathbf{F}_c$ and $\mathbf{R}$ are the vectors of external, contact and internal nodal forces, respectively. The main difference between FEM and the material point method follows from the fact that the state variables are traced at the material points, defined independently of the computational mesh in MPM, and at integration points connected with elements in FEM. The system of dynamic equations (3) is solved by means of the explicit time integration procedure. The details of the computations have been given in [3, 4].

4. Example

The process of silo discharge for a silo of rectangular horizontal cross-section with convergent hopper has been analysed. The silo geometry is shown in Fig. 1. Computations have been made with the following values of dimensions indicated in the figure: $A = 3\,\mathrm{m}$, $B = 4\,\mathrm{m}$, $a = b = 1\,\mathrm{m}$, $h_1 = h_2 = 1\,\mathrm{m}$. The following material data have been set: mass density, $\varrho = 1800\,\mathrm{kg/m^3}$, $E = 1 \cdot 10^6\,\mathrm{Pa}$, $\nu = 0.3$, $\varphi = 30^\circ$, $c = 0\,\mathrm{Pa}$, $\gamma = 50\,\mathrm{s^{-1}}$, $N = 1$. It has been assumed that the silo walls are smooth. As the silo has two symmetry planes, only one its quarter has been considered in the calculations.

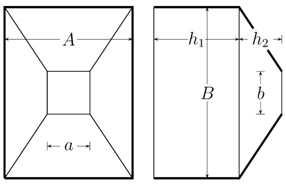

Figure 1: Silo with convergent hopper

The initial fields of displacements and stresses caused by the gravitational forces have been found solving the quasi-static problem for the silo with the closed outlet as a dynamic problem using the method of dynamic relaxation. This approach has allowed to avoid solution of a huge system of non-linear equations. A small value of viscosity parameter $\gamma = 5\,\mathrm{s^{-1}}$ and an extra viscosity term in the pressure rate–volumetric strain relation, $\dot{p} = -K\,d_{kk} - \mu\,d_{kk}$ with $\mu = 1 \cdot 10^5\,\mathrm{Pa{\cdot}s}$, have been applied in the relaxation procedure.

To discretise one quarter of the analysed region, 6052 nodes and 26652 tetrahedral elements with linear interpolation functions have been defined in the computational mesh. The bulk material has been modelled by 221608 material points. Using the explicit time integration procedure, the time increment has been set as $5 \cdot 10^{-4}\,\mathrm{s}$ for $t \le 14.7\,\mathrm{s}$ and $1 \cdot 10^{-4}\,\mathrm{s}$ for $t > 14.7\,\mathrm{s}$.

Several phases of the flow process are shown in Fig. 2. The mass flow has been observed in the silo. Significant distortions of the material could be seen in the figure.

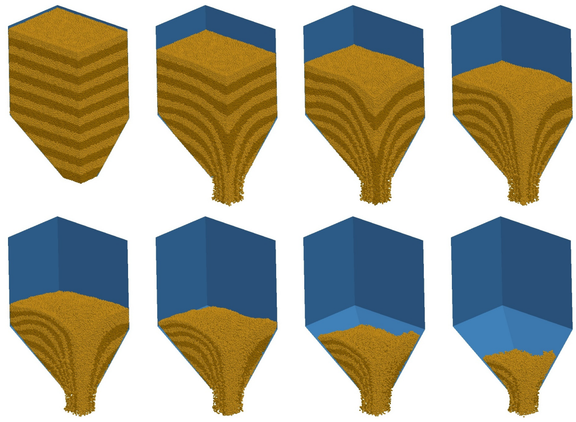

Figure 2: Several phases of silo discharge process for time values: 0 (static), 3, 5, 7, 9, 11, 13, 14.5 s

The flow rate for the bulk material has been calculated and compared to the value following from the empirical formula by Beverloo *et al.* [1]

$$W = 1.0444\,\varrho\,(a - kd)\,(b - kd)\sqrt{g\,\frac{(a - kd)\,(b - kd)}{a + b - 2kd}}$$

where g is the gravity acceleration, d grain diameter and $k \ge 1$. The comparison is illustrated in Fig. 3. Quick stabilisation of the flow rate has been obtained in the calculations as observed in the experiment [1]. The value of the calculated flow rate matches fairly well the empirical value.

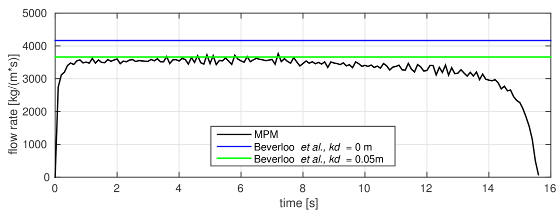

Figure 3: Flow rate

5. Conclusions

A three-dimensional flow of a bulk material during an entire silo discharge process has been analysed successfully by the material point method. The calculated flow rate of the granular material has shown very good agreement with the experimental results related to its value and time distribution.

References

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