Kansa method for problems with multiple boundary conditions

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Abstract

In the paper a kind of meshless discretization technique, called Kansa method, is investigated in the context of problems with multiple boundary conditions. This numerical method uses interpolant composed of radial basis functions as well as collocation technique to discretize differential equations. To overcome the problem that appears for equations with multiple boundary conditions, where more equations should be associated with a boundary node than degrees of freedom that exist at this node, an extension of the method is proposed. The key idea lies in the modification of the interpolant with the use of Hermite formulation. The details of the approach are shown in the paper. Moreover, the special attention is paid to estimate respective value of the shape parameter included in radial functions to ensure stability of the solution process and high accuracy. To illustrate usefulness, accuracy and convergence of the method, it is employed to solve a test problem of bending Kirchhoff plates.

Keywords: meshless methods, radial basis functions, Kansa method, Hermite interpolation, shape parameter

1. Introduction

In 1990 Kansa applied an interpolant composed of radial basis functions (RBF) in conjunction with collocation procedure to solve differential equations [1,2]. In this way, a nice meshless features of these functions, reported by Hardy [3] and Buhmann [4], were introduced into the area of differential equations. Since then, the method has attracted more and more researchers' interest due to its simplicity, rapid convergence and high accuracy. There are many papers devoted to this method and its applications [5]. In most of them the method is used to solve lower order equations possessing one boundary condition associated with a boundary. A problem arises, when one applies the method to solve differential equation of higher order. To ensure this problem to be well-posed more than one boundary condition have to be introduced on each boundary. Collocation technique applied to such a problem associates these boundary conditions in a discrete form with a boundary node, where only one unknown function value is searched for. Therefore the direct use of the method leads to overdetermined set of algebraic equations, similarly as in RBF-based pseudospectral approach, described in [6]. The solution in least squares sense can be achieved in such a case. To preserve interpolation character of the method, in the present paper, the interpolant that approximates the sought function is modified according to Hermite idea. Additional terms associated with additional degrees of freedom at boundary nodes are introduced, similarly as in another RBF collocation approach [7]. It allows the method to be easily applied in problems with multiple boundary conditions and increase accuracy of the approach.

2. Modified Kansa method

Let us consider a boundary value problem with multiple boundary conditions written in a general form as

$$Lu = f \quad \text{in } \Omega \tag{1}$$

$$B_1 u = g_1,\ B_2 u = g_2 \quad \text{on } \partial\Omega \tag{2}$$

where L, B_1, B_2 are differential operators, u denotes the sought function and f, g_1, g_2 represent known functions. Without loss of generality it is assumed that two boundary conditions are needed to make Eq. (1) well-posed.

The solution of the problem in the Kansa method is searched in the form of interpolation function composed of RBF [1,2]. In the present approach the interpolant is extended by adding some additional terms connected with boundary conditions. The modified interpolation function assumes the form

$$\begin{aligned} u(\mathbf{x}) = \sum_{j=1}^{N^I} \alpha_j \varphi\left(\left\|\mathbf{x}-\boldsymbol{\xi}\right\|\right)\Big|_{\boldsymbol{\xi}=\mathbf{x}_j^I} + \sum_{j=1}^{N^B} \beta_j \left[B_1^{\boldsymbol{\xi}} \varphi\left(\left\|\mathbf{x}-\boldsymbol{\xi}\right\|\right)\right]_{\boldsymbol{\xi}=\mathbf{x}_j^B} + \\ + \sum_{j=1}^{N^B} \gamma_j \left[B_2^{\boldsymbol{\xi}} \varphi\left(\left\|\mathbf{x}-\boldsymbol{\xi}\right\|\right)\right]_{\boldsymbol{\xi}=\mathbf{x}_j^B} \end{aligned} \tag{3}$$

where $\varphi(\|\cdot\|)$ denotes RBF, $\boldsymbol{\xi} \in \mathbb{R}^n$ are some special points called centers, which coincide with nodes $\mathbf{x}_i$, $i = 1,\ldots,N$. Among them one can distinguish interior nodes $\mathbf{x}_i^I, i = 1,\ldots, N^I$ and nodes imposed on the boundary $\mathbf{x}_i^B, i = 1,\ldots, N^B$. In Eq.(3), B_1^{ξ} and B_2^{ξ} are differential operators from boundary conditions acting on the radial function treated as a function of $\boldsymbol{\xi}$ variable.

Introducing Eq. (3) into Eqs (1)-(2) and by collocating it at each node, the system of equations for unknown interpolation coefficients α_i, β_i, γ_i is obtained. Its solution can be written as follows

$$\begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_{L^x} & \boldsymbol{\Phi}_{L^x B_1^{\xi}} & \boldsymbol{\Phi}_{L^x B_2^{\xi}} \\ \boldsymbol{\Phi}_{B_1^x} & \boldsymbol{\Phi}_{B_1^x B_1^{\xi}} & \boldsymbol{\Phi}_{B_1^x B_2^{\xi}} \\ \boldsymbol{\Phi}_{B_2^x} & \boldsymbol{\Phi}_{B_2^x B_1^{\xi}} & \boldsymbol{\Phi}_{B_2^x B_2^{\xi}} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathbf{f} \\ \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix} \tag{4}$$

where the entries of the block matrices included in coefficient matrix in Eq. (4) follow from the imposition of the appropriate differential operators on interpolant (3) and evaluation of the obtained expressions at respective nodes.

Once the interpolation coefficients are obtained the solution can be described at any point by Eq. (3).

2.1. Remarks on the solvability and stability of the solution process

The unique solution (Eq. (4)) exists if only the coefficient matrix is invertible. Some remarks on the invertibility of the matrix, in the case of the original Kansa method, can be found in various papers [8]. From this information one can conclude that although strictly positive definite RBF are used or interpolant (3) is augmented by polynomial terms, the Kansa matrix can be singular. But these cases are very rare and there are many examples of successful application of the method. Based on findings from [5], the same conclusion can be drawn for the modified Kansa method. To ensure the matrix to be invertible, Fasshauer [5] introduced the differential operator from the governing equation into the interpolant. It makes the Kansa matrix be symmetric one, and therefore invertible. This approach can be also incorporated into the modified approach presented in the present paper.

Another issue connected with obtaining solution (4) is the stability of the solution process. To achieve high accuracy one should adjust the value of the shape parameter included in RBF. The values that theoretically lead to high accuracy make the matrix ill-conditioned and therefore the system difficult to solve. To obtain appropriate value of the parameter a heuristics that relates the accuracy and stability with a number of significant digits assumed for computation is used. The details are shown in [6].

3. Test problem

As a benchmark problem, the method has been applied to solve the static problem of thin, isotropic quadrilateral plates governed by the equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \qquad (5)$$

where w denotes the transverse plate displacement, q is the applied lateral force per a unit of area and D is the flexural rigidity of the plate.

In present paper, combination of simply-supported and clamped boundary conditions are considered. Noting them in terms of operators used in Eq. (3) one can put

$$B_1 w = 0,\ B_2 w = 0 \qquad (6)$$

where B_1=1 and

$$B_2 = \left(\cos^2\theta + \nu\sin^2\theta\right)\frac{\partial^2}{\partial x^2} + \left(\sin^2\theta + \nu\cos^2\theta\right)\frac{\partial^2}{\partial y^2} + 2(1-\nu)\cos\theta\sin\theta\frac{\partial^2}{\partial x \partial y} \quad \text{for simply-suported edge}$$

or $B_2 = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$ for clamped edge

In above equations θ is the angle between the normal to the plate boundary and the *x*-axis.

The problem has been solved by the method described in section 2 using multiquadric RBF and some results for the square plates are presented in Tab. 1. For comparison purposes, in the table the reference results obtained by Levy's (single series) method and finite difference one are also included. Both uniform and irregular distributions of nodes have been tested.

Table 1 shows good agreement of the results obtained by the modified Kansa method with the reference results.

Table 1: Normalized maximum deflection of the square plate

	SSSS	CCCC	SCSC
uniform grid			
N=121 N^I = 81	4.032e-3	1.262 e-3	1.907e-3
N=225 N^I = 169	4.051e-3	1.263 e-3	1.912e-3
irregular grid			
N=121 N^I = 81	4.060e-3	1.263 e-3	1.915e-3
N=225 N^I = 169	4.058e-3	1.262 e-3	1.914 e-3
Reference results	**4.062e-3**	**1.265e-3**	**1.917 e-3**

4. Conclusion

In the paper the use of a kind of collocation method that takes advantage of RBF interpolation to solve differential equations of higher order is shown. Equations of this type possess multiple boundary conditions and the direct use of the collocation techniques is cumbersome. To enable the Kansa method to be applied to such problems the interpolation function has been extended according to Hermite interpolation idea. To this end differential operators from boundary conditions have been included in the interpolant. The method leads to a system of equations with non-symmetric coefficient matrix. By imposing the differential operator from governing equation on some terms of the interpolant this matrix can be assembled as a symmetric one. In the paper the method has been validated by the problem of bending of thin plates. By choosing appropriate value of the shape parameter in RBF the method is able to provide very accurate results.

References

[1] Kansa, E., Multiquadrics – A scattered data approximation scheme with applications to computational fluid dynamics I: Surface approximations and partial derivative estimates, *Computers Math. Applic.*, 19, pp. 127-145, 1990.

[2] Kansa, E., Multiquadrics – A scattered data approximation scheme with applications to computational fluid dynamics II: Solutions to parabolic, hyperbolic, and elliptic partial differential equations, *Computers Math. Applic.*, 19, pp. 147-161, 1990.

[3] Hardy, R.L., Multiquadric equations of topography and other irregular surfaces, *J. Geophysics Res.*, 176, pp. 1905-1915, 1971.

[4] Buhmann, M.D., *Radial basis functions*, Cambridge University Press, New York, 2000.

[5] Fasshauer, G.E., *Meshfree Approximation Methods with Matlab*, World Scientific Publishing, Singapore, 2007.

[6] Krowiak, A., Radial basis function-based pseudospectral method for static analysis of thin plates, *Engineering Analysis with Boundary Elements*, 71, pp. 50-58, 2016.

[7] Krowiak, A., Hermite type radial basis function-based differential quadrature method for higher order equations, *Applied Mathematical Modelling*, 40, pp. 2421-2430, 2016.

[8] Chen, W., Fu, Z.J., Chen, C.S., *Recent Advances in Radial Basis Function Collocation Methods*, Springer, 2014.