Equilibrium based finite element method in axi-symmetric problems

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Abstract

A stress based formulation of the finite element method has been applied to the axially symmetric equilibrium problem for an elastic solid. The case of fully or nearly incompressible material has been taken into account. A rectangular element has been proposed to approximate statically admissible stress fields. The obtained results have been compared with outcomes gained by the u–p (mixed) method.

Keywords: stress-based finite element method, volumetric locking, C^1 elements, stress function

1. Introduction

The isochoric deformation is not embedded in the approximation of the deformation field by means of linear or quadratic interpolation function used commonly in the displacement-based finite element method (FEM). As a result, in an analysis of an isotropic material with Poisson's ratio close to 0.5, a phenomenon of volumetric locking is observed. There are some ways of alleviating the influence of locking like averaging (smoothing) the volumetric strain or the pressure or the use of the mixed formulation, u–p, formulation where, beside the displacement field, the pressure field is a primary unknown, e.g. [1]. The stress based formulation of the finite element method allows one to avoid the difficulties related to locking phenomenon. This formulation is based on the principle of complementary work. However, it is rather rarely used in the research and engineering (except the case of torsion problems). The main reason seems to be a harder way of construction of statically admissible stress fields in comparison to the interpolation of kinematically admissible displacement fields in the displacement-based FEM.

In the present paper, the stress fields in the axially symmetric problem have been constructed by means of stress function having two components F and M. The first component has been interpolated by use of element with shape functions of C^1 class while the second one by means of shape functions of C^1 class with respect to the radial coordinate and C^0 class with respect to the vertical one.

2. Problem statement

A static, small deformation problem for an isotropic elastic body is analysed. No restriction for the value of Poisson's ratio is assumed; it may be close to 0.5 or equal to this number. In the considered axially symmetric case, the four non-zero components of the stress field satisfy the following two equilibrium equations:

$$\begin{aligned} \sigma_{rr,r} + \sigma_{rz,z} + (\sigma_{rr} - \sigma_{\varphi\varphi})/r = 0 \\ \sigma_{rz,r} + \sigma_{zz,z} + \sigma_{rz}/r = 0 \end{aligned} \tag{1}$$

where r, z and φ denote the cylindrical coordinates while the comma indicates the partial derivative. Physical components of the stress tensor have been used in Eq. (1), and $\sigma_{r\varphi} = \sigma_{z\varphi} = 0$.

Let us define the set of statically admissible fields of stresses:

$$\begin{aligned} Y = \{\boldsymbol{\tau} \in [L^2(\Omega)]^9 : \tau_{ij} = \tau_{ji},\ \boldsymbol{\tau} \text{ satisfies Eq. (1) in } \Omega, \\ \tau_{\alpha\beta}\, n_\beta = t_\alpha \text{ on } \Gamma_\sigma\} \end{aligned} \tag{2}$$

where Ω denotes the section in rz plane of the region occupied by the analysed body, n_α the vector outwardly normal to the boundary of Ω, Γ_σ the part of the section where the Cauchy stress vector, t_α, is given, $x_1 = r$, $x_2 = z$, $\alpha = 1, 2$, and L^2 denotes the space of square-integrable functions.

The static problem for the considered axisymmetric elastic body can be stated as follows: Find the stress field $\boldsymbol{\sigma} \in Y$ such that the variational equation holds

$$\int_\Omega C_{ijkl}\, \sigma_{kl}\, (\tau_{ij} - \sigma_{ij})\, \mathrm{d}x = \int_{\Gamma_u} U_i\, (\tau_{ij} - \sigma_{ij})\, n_j\, \mathrm{d}s \qquad \forall \boldsymbol{\tau} \in Y \tag{3}$$

where C_{ijkl} denotes the tensor of elastic compliances and Γ_u the part of the boundary of section Ω where displacements U_i are given. Eq. (3) expresses the principle of complementary work.

3. Finite element solution

In order to satisfy the equilibrium equations (1) inside region Ω, a stress function with two components F (known as Love's stress function) and M has been applied. The non-zero stress components are expressed by means of the stress function as follows [3]:

$$\begin{aligned} \sigma_{rr} &= F_{,zz} + M, & \sigma_{zz} &= F_{,rr} + F_{,r}/r, \\ \sigma_{\varphi\varphi} &= (r\, M)_{,r} + F_{,zz}, & \sigma_{rz} &= -F_{,rz}. \end{aligned} \tag{4}$$

To obtain convergence of the finite element solution, it is required to use interpolation functions of class C^1 for function F and of class C^0 for function M. This follows from the orders of derivatives of F and M appearing in the equation of complementary

work (3). Thus, a rectangular element with shape functions defined as follows: $F \in Q_3(r) \times Q_3(z)$ and $M \in Q_1(r) \times Q_1(z)$ can be used where $Q_n(x)$ denotes the space of polynomials of nth degree with respect to variable x. This means that, at each node $\boldsymbol{x}_i$, four degrees of freedom are used for function F: $F(\boldsymbol{x}_i)$, $\frac{\partial F}{\partial r}(\boldsymbol{x}_i)$, $\frac{\partial F}{\partial z}(\boldsymbol{x}_i)$ and $\frac{\partial^2 F}{\partial r \partial z}(\boldsymbol{x}_i)$, and one degree of freedom for function M: $M(\boldsymbol{x}_i)$. However, such a choice of shape functions leads, to non-smooth distribution of stress component $\sigma_{\varphi\varphi}$ as the degrees of polynomials in the case of this component generated by the two stress function components are different. Therefore, it has been proposed to apply stress function M as an element of space $Q_3(r) \times Q_1(z)$ which means that two degrees of freedom are defined for M at each node: $M(\boldsymbol{x}_i)$ and $\frac{\partial M}{\partial r}(\boldsymbol{x}_i)$. The proposed finite element has 24 degrees of freedom and satisfies exactly equilibrium equations (1) at any point of region Ω. To fulfill the equilibrium conditions on boundary Γ_σ, the Lagrange multiplier method is employed [4].

4. Numerical example

A thick circular plate loaded uniformly on its top surface and clamped on its cylindrical surface has been considered. The thickness and radius of the plate has been assumed 1 m and 1.5 m, respectively, while the load intensity 10 Pa. Calculations have been made for the following material properties: Young's modulus $E = 1$ GPa and Poisson's ratio $\nu = 0.49999$. The results of the calculations based on the proposed stress-based formulation have been compared to the outcomes of the mixed (u–p) finite element approach. The value of Poisson's ratio assumed in the calculations is the largest acceptable value in the ANSYS finite element software [2] which has been utilised to get the u–p solution. The eight node rectangular element has been used in the mixed approach with parabolic interpolation functions and the linear polynomial to approximate the fields of the displacement vector and the pressure in the area of the element, respectively.

The results of calculations are shown in figures 1–4 in the form of colour maps for all four stress components: σ_{rr}, $\sigma_{\varphi\varphi}$, σ_{zz} and σ_{rz}. In the diagrams, the outcomes are illustrated for cross-section $\varphi = \text{const}$, $r \in [0, 1.5]$ m, $z \in [0, 1]$ m. The computational grids – displayed in the figures – with 2×4 elements have been applied in both the approaches. As seen in the plots, the stress field obtained by the use of the present equilibrium approach is much smoother than that calculated by the mixed method. In the case of the proposed approach, more significant roughness of the stress field has been observed only for the region located in the vicinity of the clamped boundary.

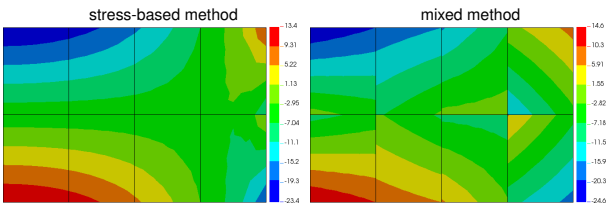

Figure 1: Radial normal stress component, σ_{rr} [Pa]

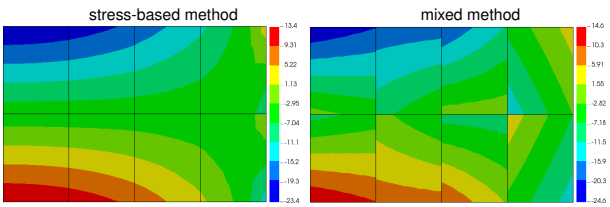

Figure 2: Circumferential normal stress component, $\sigma_{\varphi\varphi}$ [Pa]

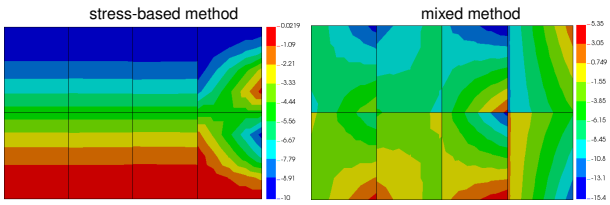

Figure 3: Vertical normal stress component, σ_{zz} [Pa]

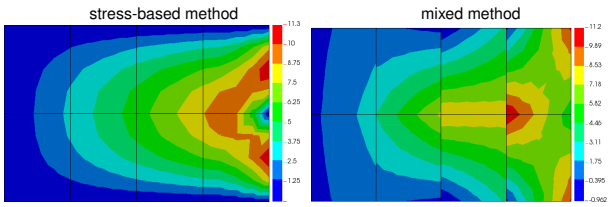

Figure 4: Tangential stress component, σ_{rz} [Pa]

The results related to a denser computational mesh with 4×8 elements are illustrated in Fig. 5 for the circumferential normal stress component. Quite remarkable discontinuities have been seen in the case of the mixed method while the stress-based method has produced very smooth result even with the use of the coarse mesh.

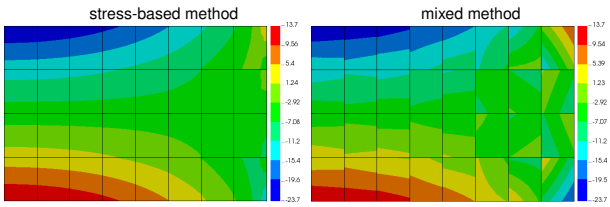

Figure 5: Circumferential normal stress component, $\sigma_{\varphi\varphi}$ [Pa], denser mesh case

5. Concluding remarks

The stress-based model of the finite element method has been applied to the axially symmetric equilibrium problem for the linearly elastic material. The proposed rectangular element with 24 degrees of freedom has shown convergence of the approximate solution. The proposed approach allows to find smoother and more accurate stress field than the mixed displacement–pressure formulation of FEM utilised commonly in computational mechanics.

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