# Analytical approach to determine vertical dynamics of a semi-trailer truck from the point of view of goods protection 

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#### Abstract

This paper presents an analytical method to calculate the vertical dynamics of a semi-trailer truck, containing general viscous damping and exposed to harmonic base excitation. For the purpose of a clearer understanding, the method will be presented through a simplified four degrees-of-freedom half-vehicle model, which neglects the stiffness and damping of the tyres, thus the four degrees-of-freedom are the vertical and angular displacement of the truck and the trailer. From the vertical and angular acceleration of the trailer, the vertical acceleration of each point of the platform of the trailer can easily be determined, from which the forces acting on the transported goods are given.


Keywords: semi-trailer truck, vertical dynamics, vibration, goods protection, transfer-function

## 1. Introduction

One useful application of vehicle simulation is the estimation of the vertical vibration, which is widely accepted as being directly related to product damage during transportation according to Ref. [3]. To get the motion characteristics of the vehicle, the equation of motion of the vehicle - modelled as a multi degree-of-freedom vibration system - has to be solved. Basically there are two ways to solve this second order differential equation, one way is to use a step-by-step numerical method as done in Ref. [2], while the other way is to analytically solve the equation, as demonstrated in Ref. [4] for a full car model. With the former method, one can get an approximate solution in time domain caused by arbitrary base excitation. Using the latter way, one can get an exact solution of the steady state response caused by a harmonic base excitation in frequency domain. With this, it can easily be observed which frequency is the most dangerous and what accelerations occur at this frequency. In this paper the latter method will be discussed in details through a simplified four degrees-of-freedom model of a semi-trailer truck.

## 2. The mechanical model

Figure 1 shows the mechanical model of the semi-trailer truck under discussion.


Figure 1: Mechanical model of the semi-trailer truck
The model has four degrees-of-freedom, with the following general displacements: $q_{1}$ : vertical displacement of the truck $q_{2}$ : angular displacement of the truck $q_{3}$ : vertical displacement of the trailer $q_{4}$ : angular displacement of the trailer. The
equation of motion of the system with matrix notations looks as follows:

where $\underline{\underline{M}}, \underline{\underline{C}}$ and $\underline{\underline{K}}$ are the mass, damping and stiffness matrices respectively, while $\stackrel{V}{=}$ and $\underset{\underline{W}}{ }$ are the coefficient matrices of the base excitation and its first derivate.

## 3. Analytical solution of the equation of motion

According to Ref. [1], transforming Eqn (1) time domain matrix equation into the Laplace domain (with variable $s=\sigma+j \omega$ ), yields:
$(\underbrace{\left(\underline{M} s^{2}\right.}_{\underline{\underline{Z}}(s)}+\underset{\underline{C}}{=}+\underline{\underline{\underline{K}}}) \underline{\underline{Q}}(s)=(\underline{\underline{W} s}+\underset{=}{\underline{V}}) \underset{=}{\underline{Q}}(s)$,
where $\underline{\underline{Z}}(s)$ is the dynamic stiffness matrix.
Inverting Eqn (2) results in the definition of the transfer function:
$\underline{\underline{H}}(s)=\underline{\underline{Z}}^{-1}(s)(\underline{\underline{W} s}+\underline{\underline{V}})=\frac{\operatorname{adj}(\underline{\underline{Z}}(s))(\underline{\underline{W} s}+\underline{\underline{V}})}{|\underline{\underline{Z}}(s)|}$.
The denominator of Eqn (3) is the system characteristic equation, the roots of which are the system poles, which define the resonance frequencies of the system. To solve this eigenvalue problem, Eqn (3) has to be transformed into state space form shown in Eqn (4).

The general eigenvalue problem of the above equation generates $2 k$ ( $k=$ number of $D O F$ ) complex valued eigenvalues, appearing in complex conjugate pairs. Thus the spectral and modal matrix looks as follows:
$\underline{\underline{\Lambda}}=\left\langle\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}, \lambda_{1}^{*}, \lambda_{2}^{*}, \lambda_{3}^{*}, \lambda_{4}^{*}\right\rangle$,
$\Psi=\left[\begin{array}{ccc:ccc}\lambda_{1} \psi_{1} & \cdots & \lambda_{4} \psi_{4} & \lambda_{1}^{*} \psi_{1}^{*} & \cdots & \lambda_{4}^{*} \psi_{4}^{*} \\ \hdashline--y_{4} & \cdots & \psi_{4} & \psi_{1}^{*} & \cdots & \psi_{4}^{*}\end{array}\right]$,
where $\rangle$ notates the diagonal elements of the spectral matrix.
To solve Eqn (4) state space equation, the equations have to be decoupled. As the modal vectors are orthogonal with respect both to $\underset{=}{A}$ and $\underset{=}{B}$ matrices, a transformation with (6) modal matrix leads to uncoupled equations. This transformation is summarized in Eqn (7).

From this, the state vector in modal space can be calculated as

After back-transformation to physical space, the state vector is
$\underline{\underline{U}}(s)=\underline{\underline{\Psi}}(s I=-\underline{\underline{\Lambda}})^{-1} \stackrel{\Theta}{\underline{\Theta}}^{-1} \underline{\underline{\Psi}}^{T} \underline{\underline{G}} \underline{=}(s)$.
To gain the transfer function describing the response due to unit base excitation, an excitation matrix has to be substituted into Eqn (9), the columns of which contain the unit base excitations belonging to each wheel, shown in (11)
$\underline{\underline{H}}(s)=\underline{\underline{\Psi}}(s \underline{=}-\underline{\underline{\Lambda}})^{-1} \stackrel{\Theta}{ }^{-1} \underline{\underline{\Psi^{T}}} \underline{\underline{R}}$,

As a simplification $\Gamma_{i j}$ notation was introduced, where:
$\Gamma_{i j}=\sum_{k=1}^{4} \psi_{k i}\left(W_{k j} s+V_{k j}\right)$.
Using this notation, the transfer function looks as follows:
$\underline{\underline{H}}(s)=\underline{\underline{\Psi}}(s \underline{=}-\underline{\underline{\Lambda}})^{-1} \underline{\underline{\Theta}}^{-1} \underline{\underline{\Gamma}}$.
Expanding the above matrix and utilizing the modal superposition theorem, the transfer function belonging to the unit base excitation acting on the $j$ th wheel and the response on the $k$ th general displacement looks as follows:

$$
\begin{equation*}
H_{k j}(s)=\sum_{i=1}^{4}\left(\frac{\Psi_{k i} \vartheta_{i}^{-1} \Gamma_{i j}}{s-\lambda_{i}}+\frac{\Psi_{k i}^{*} \vartheta_{i}^{*-1} \Gamma_{i j}^{*}}{s-\lambda_{i}^{*}}\right) \tag{14}
\end{equation*}
$$

## 4. Numerical example

To visually represent the formulas derived in the previous chapter, the representative values for the parameters of the system shown in Table 1. have been used.

Table 1: Representative values of the simulation

| $m_{1}=6000 \mathrm{~kg}$ | $k_{4}=10^{6} \mathrm{~N} / \mathrm{m}$ | $l_{1}=2 \mathrm{~m}$ |
| :--- | :--- | :--- |
| $I_{2}=10^{5} \mathrm{kgm}^{2}$ | $k_{5}=5000 \mathrm{Nm}$ | $l_{2}=2 \mathrm{~m}$ |
| $m_{3}=20000 \mathrm{~kg}$ | $c_{1}=c_{2}=c_{3}=10^{3} \mathrm{Ns} / \mathrm{m}$ | $l_{3}=1.8 \mathrm{~m}$ |
| $I_{4}=350000 \mathrm{kgm}^{2}$ | $c_{4}=10^{3} \mathrm{Ns} / \mathrm{m}$ | $l_{4}=5 \mathrm{~m}$ |
| $k_{1}=k_{2}=k_{3}=2 \cdot 10^{5} \mathrm{~N} / \mathrm{m}$ | $c_{5}=5 \cdot 10^{3} \mathrm{Nsm}$ | $l_{5}=5 \mathrm{~m}$ |

The frequency response function $(F R F)$ is a subset of the transfer function, it is the cross section along the frequency $(\omega)$ axis. To get the $F R F$ between base excitations acting on all three wheels and the one of the general displacements, the complex response vectors - belonging to each base excitation have to be added. During addition, the phase shifts between the base excitations can also be taken into account, by rotating the complex vectors with the appropriate phase angle. Figure 2 represents the $F R F$ s between base excitations acting an all three wheels - with same phase - and the vertical and angular displacement and acceleration of the trailer.


Figure 2: Frequency response functions of the trailer
Knowing the vertical and angular acceleration of the trailer, the maximum acceleration on each point of the platform can easily be calculated. Figure 3 represents the peak accelerations in function of the distance from the trailer's center of gravity, applying different excitation frequencies. Note that the frequency of $5 \mathrm{rad} / \mathrm{s}$ and $20 \mathrm{rad} / \mathrm{s}$ are plotted on a secondary axis for better visibility.


Figure 3: Peak accelerations of the trailer

## References

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