Local discontinuous Galerkin method with arbitrary polygonal finite elements

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Abstract

The paper deals with the so-called local discontinuous Galerkin (LDG) method. The LDG method was developed almost twenty years ago and is now well established in numerical analysis of engineering problems. The crucial point of the method is related to the so-called numerical fluxes that have to be evaluated on the mesh skeleton. In this paper, an alternative approach for the numerical fluxes is proposed, based on finite difference relations. Thus, in this paper the method is called LDG with finite difference (LDGFD). The LDGFD method is presented for two-dimensional non-stationary heat transport. It has been demonstrated that the LDGFD method is correct and stable, and it may be successfully applied to meshes with arbitrary polygonal finite elements. Where finite elements are rectangular, very high order approximations may be achieved with Chebyshev polynomials used as the basis functions. The work is illustrated with various examples presenting the advantages of LDGFD method.
Keywords: local discontinuous Galerkin method, Chebyshev polynomials, finite difference

1. Introduction

Discontinuous Galerkin (DG) methods have become popular mainly due to their computational flexibility and efficiency. Nevertheless, only one DG method âĂŞ of the many existing ones âĂŞ falls within the scope of interest of this work âĂŞ the so-called local DG (LDG) method [1, 2]. The main idea of the LDG method is to write the original problem into a system of first order equations. The weak form system of equations is subsequently constructed taking into account discontinuous approximations. The key components of the method are the mesh skeleton numerical fluxes that have to be set. Well designed numerical fluxes guarantee stability of the method. There is no one formula for numerical fluxes in the LGD method, and they have the form of empirical relations in which some parameters have to be set by the user.

In this paper, an alternative approach to numerical fluxes is proposed. In this approach, one dimensional finite difference (FD) relations are used for evaluations of required fluxes on the mesh skeleton. A similar approach has been applied to enforce boundary conditions of both Dirichlet and Neumann types. An analogous method, also using FD relations, was successfully applied and presented in AuthorâĂŹs previous papers, e.g. [3, 4], which were discussing the standard DG method. However, the LDG method requires the use of more sophisticated finite difference relations. For the sake of clarity, the version of the LDG method presented in this work is called LDG with finite difference rules (LDGFD) to distinguish it from the standard LDG method.

The LDGFD method is applied here to a two-dimensional (2D) scalar parabolic problem which has the physical interpretation of non-stationary heat flow. In this problem, the main field is temperature and the heat flux vector refers to the temperature gradient. In the LDGFD method, both fields are approximated. In this work, the LDGFD method is applied to polygonal meshes that refer to the cases in which the finite element cells have arbitrary shapes and may be convex or non-convex. Here, the cells may be non-simply-connected (NSC) (i.e. they may have holes inside) or even non-connected (NC) (i.e. consisting of two or more parts which are completely separated from each other). The approximation in the finite element cells is based on a set of basis functions which may consist of quite arbitrary functions. It means that there are no special requirements for those functions on edges or vertices of the cell. For example, the basis functions may be just monomials or other polynomials, such as Chebyshev and Legendre. When the polygonal cells are rectangular, it is possible to get very high-ordered approximation with $p > 10$, or even $p = 50$.

A series of examples have been presented in this work to illustrate the LDGFD method, including, in some of them, a comparison with the standard LDG method. In a benchmark example, a convergence analysis is presented for both the LDGFD and LDG methods. In the other example, the problem with heat flux concentration is considered, and the results are compared with those obtained by the standard DG method.

The work deals with discontinuous approximations, thus the following operators need to be defined

$$[\![f]\!] = \lim_{\epsilon\to 0} [\![f]\!]_\epsilon\,, \quad \langle f\rangle = \lim_{\epsilon\to 0} \langle f\rangle_\epsilon \tag{1}$$

where the jump and mean values at distance ϵ are as follows:

$$[\![f]\!]_\epsilon = f(\mathbf{x}+\epsilon\,\mathbf{n}) - f(\mathbf{x}-\epsilon\,\mathbf{n}) \tag{2}$$

$$\langle f\rangle_\epsilon = 0.5\Big(f(\mathbf{x}+\epsilon\,\mathbf{n}) + f(\mathbf{x}-\epsilon\,\mathbf{n})\Big) \tag{3}$$

2. Mathematical model

In this work, a non-stationary heat flow in a 2D domain is going to be analysed. However, for the sake of simplicity, a problem related to a stationary case has been selected here to present the proposed approach, which reads as:

$$\begin{aligned} &\operatorname{div}\mathbf{q} - r = 0\,, \quad \mathbf{q} = -\lambda\nabla t \qquad \text{in } V \\ &t = \hat{t} \quad \text{on } S_t\,, \quad \mathbf{q}\cdot\mathbf{n} = \hat{q} \quad \text{on } S_q \end{aligned} \tag{4}$$

where $\mathbf{q}$ is the heat flux vector, r the heat source density, λ is the heat conductivity parameter for a thermally isotropic material, $\hat{t}$ and $\hat{q}$ are prescribed values of temperature and heat flux, respectively, S_t is the part of S where the temperature $\hat{t}$ is prescribed, S_q is the part of S where the heat flux $\hat{q}$ is prescribed, $\mathbf{n}$ is the unit vector normal to the outer boundary.

It may be seen that the first two relations in eq. (4) can be expressed by two combined equations in the following weak forms with test functions v and $\boldsymbol{\tau}$ that take into account both kinds of boundary conditions:

$$
\begin{aligned}
&\int_{S_q} v\hat{q}\,\mathrm{d}S + \int_{S_t} v\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}S + \int_{S_s} [\![v]\!]\mathbf{q}\cdot\mathbf{n}^s\,\mathrm{d}S \\
&- \int_V \nabla v\cdot\mathbf{q}\,\mathrm{d}V - \int_V vr\,\mathrm{d}V = 0 \quad \forall\, v
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
&\int_{S_q} \lambda\boldsymbol{\tau}\cdot\mathbf{n}\,t\,\mathrm{d}S + \int_{S_t} \lambda\boldsymbol{\tau}\cdot\mathbf{n}\,\hat{t}\,\mathrm{d}S \int_{S_s} \lambda[\![\boldsymbol{\tau}]\!]\cdot\mathbf{n}^s\,t\,\mathrm{d}S \\
&- \int_V \mathrm{div}(\lambda\boldsymbol{\tau})\,t\,\mathrm{d}V + \int_V \boldsymbol{\tau}\cdot\mathbf{q}\,\mathrm{d}V = 0 \quad \forall\,\boldsymbol{\tau}
\end{aligned}
\tag{6}
$$

where $\mathbf{n}^s$ is the unit vector normal to the mesh skeleton.

In eqs. (5) and (6) the t and $\mathbf{q}$ have to be additionally evaluated on S_s and on S_q and S_t, respectively. In the standard LDG, they are substituted by the so-called numerical fluxes, but LDGFD stipulates that special one-dimensional finite difference rules are to be prepared for such values.

3. Approximations

In the LDGFD method, both the temperature and heat flux vectors are approximated, and so are the test functions v and $\boldsymbol{\tau}$. The approximations of t and $\mathbf{q}$ on the whole mesh may be written in the following form:

$$
t = \boldsymbol{\Phi}_t\check{\mathbf{t}}\,, \quad \mathbf{q} = \boldsymbol{\Phi}_q\check{\mathbf{q}} \qquad \text{in } V \tag{7}
$$

where $\boldsymbol{\Phi}_t$ and $\boldsymbol{\Phi}_q$ are the appropriate approximation matrices and $\check{\mathbf{t}}$, $\check{\mathbf{q}}$ are vectors of degrees of freedom for temperature and heat flux, respectively. And thus, the jumps and mean values may be obtained using the same approximation matrices and vectors of degrees of freedom:

$$
\begin{aligned}
[\![t]\!] &= [\![\boldsymbol{\Phi}_t]\!]\check{\mathbf{t}}\,, \quad \langle t\rangle = \langle\boldsymbol{\Phi}_t\rangle\,\check{\mathbf{t}} \\
[\![\mathbf{q}]\!] &= [\![\boldsymbol{\Phi}_q]\!]\check{\mathbf{q}}\,, \quad \langle\mathbf{q}\rangle = \langle\boldsymbol{\Phi}_q\rangle\,\check{\mathbf{q}}
\end{aligned}
\qquad \text{on } S_s \tag{8}
$$

Matrix $\boldsymbol{\Phi}_t$ consists of a set of basis functions defined locally for each finite element cell. The basis functions are not continuous when coming from one element to another. The order of basis functions may vary in neighbouring elements. The same refers to $\boldsymbol{\Phi}_q$; however, if the order of an element is p to approximate temperature, the approximation order for the heat flux is $p+1$ in this element. In the finite elements method, the approximation functions (Lagrange polynomials) for a single cell are strictly connected with the shape of the cell. In LDGFD there is no such connection, so the shape of the cells may be quite arbitrary.

Taking into account the approximations in eq. (7) and (8), a system of equations is obtained in eqs. (5) and (6), which has the following form:

$$
\begin{bmatrix} \mathbf{K}_{tt} & \mathbf{K}_{tq} \\ \mathbf{K}_{qt} & \mathbf{K}_{qq} \end{bmatrix}
\begin{bmatrix} \check{\mathbf{t}} \\ \check{\mathbf{q}} \end{bmatrix}
=
\begin{bmatrix} \mathbf{F}_t \\ \mathbf{F}_q \end{bmatrix}
\tag{9}
$$

4. Examples

Several examples have been included in this work. One of them presents an analysis of a domain with a square hole inside, Fig. 1a. The problem has been solved by the LDGFD method on various meshes, including the polygonal mesh shown in Fig.1b.

The results of calculations are presented in the form of maps of temperature and heat flux components in Fig. 2. In this example, the fluxes concentrations on the hole vertices are observed and they tend to infinity. In the maps, the values of the heat fluxes are limited to 100.

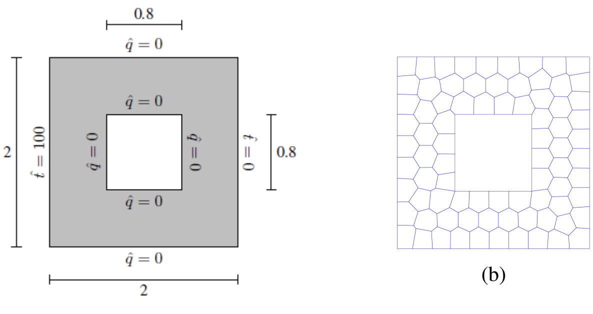
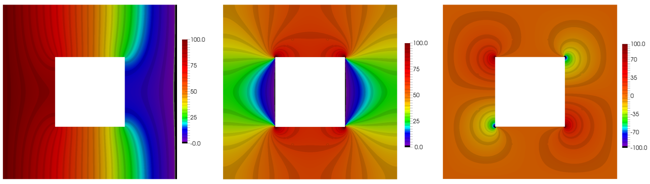


Figure 1: Domain with a hole and a polygonal mesh.



(a) t (b) q_x (c) q_y

Figure 2: Maps of temperature and heat flux components.

5. Conclusions

This work presents the LDGFD method. As the method is alternative to the standard LDG method, the paper also includes a comparison of the two methods. It has been shown that the LDGFD method is correct and stable. The LDGFD method can be applied in meshes with arbitrary polygonal finite elements. The stability of the method is particularly striking when used for high-ordered approximations, where LDG fails to converge. The *hp* mesh refinement in the LDGFD method is very easy. The mesh may be nonconforming and high order elements may be set side by side with low order elements.

References

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