Derivation of equations of the model of the dynamic behaviour of the three-dimensional atmospheric cloud of electrically charged ice crystals under the influence of electrostatic forces

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Abstract

The aim of the work is to create a theoretical model describing dynamic behaviour of electrically charged clouds of ice crystals. In the paper authors consider only the rotation of crystals under the influence of electrostatic forces, plane motion of crystals is neglected. Crystals were modelled as thin round disks to simulate hexagonal plate ice crystals, as a result, it was possible to reduce the issue to the problem with two degrees of freedom per one crystal, which enable rotation about the center of gravity of a crystal. The basic assumption of the model is statement about cloud synchronization, ie. the differences in the rotation of neighbouring crystals are small. Considering this assumption transition from equilibrium equations of moments from electrostatic forces acting on the single crystal and discrete equations of motion of the clouds to continuous differential equations describing the dynamic behaviour of such medium at the macro level was possible.

Keywords: mathematical modelling, rotating vibrations, non-linear differential equations, finite difference method, Taylor expansion

1. Introduction

The dynamic behaviour of the three-dimensional atmospheric cloud consisting of electrically charged ice crystals is analysed. The problem of modelling the dynamic behaviour of electrically charged clouds of ice crystals is a relatively new issue, which was begun with work [1], in which author presented an explanation of an optical atmospheric phenomenon called the "miracle of the sun" using synchronous vibrations of electrically charged clouds of ice crystals. Simplified, linear and one-dimensional mathematical model proposed in this paper led to solutions in the form of harmonic vibrations. Following works on this subject [2, 3] presented an increasingly complex models, taking into account non-linear and two-dimensional continuous distribution of electric charge on the surface of crystals and large amplitude of vibrations, however, a three-dimensional model, which could be used for modelling real, atmospheric clouds of ice crystals have not been established.

The objects of consideration are atmospheric clouds consisting of electrically charged ice crystals. An example of such cloud along with the global coordinate system is shown in Fig. 1. The aim is to derive and discuss a certain mathematical model for the analysis of dynamic behaviour for these clouds. Unknowns of the model are the angles describing the arrangement of crystals in space.

2. Assumptions

In order to obtain a mathematical model of dynamic behaviour of the three-dimensional atmospheric cloud, it is assumed that:

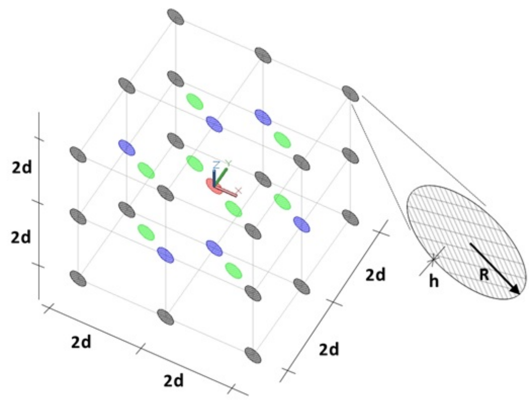

Figure 1: Analysed crystal (marked in red) with its 34 closest neighbours. Green indicates the next 8 crystals located at the corners of a cube with side length 2d and the center in the middle of the central crystal. Next 6 crystals (highlighted in blue) located at a distance of 2d from the central crystal. Another crystals are marked in black.

1. The cloud which occupies an area $\Omega \in \mathbb{R}^3$ is homogeneous, and all the crystals are identical.
2. The crystals have the shape of circular disk with a radius R and height h, wherein $h \ll R$, cf. Fig. 1.
3. The crystals are evenly distributed in all three spatial directions, cf. Fig. 1.
4. The motion of the crystals in all directions are neglected, assuming that their center of gravity does not move relative to each other.
5. Rotary motion of the crystals on its axis is neglected.
6. The electrical charge is evenly distributed on the top and bottom surfaces of each crystal.

7. We consider electrostatic forces between the analysed crystals and its closest environment (14 closest adjacent crystals), cf. Fig. 1.

8. The synchronization occurs between the crystals, ie. the differences between the angles between adjacent crystals are appropriately small.

3. Mathematical modelling

Under mentioned above assumptions we obtain a non-linear, continuous model with two degrees of freedom. The unknown of this model are 2 angles: Θ defning the deviation of a crystal from a horizontal position and Ξ defning the plane of this deflection relative to global coordinate system. Temporary local coordinate systems of crystals are assumed to be defined by the above unknowns. Local axis O'X' was taken as the direction of steepest descent in the global coordinate system, axis O'Y' was taken as the horizontal direction, cf. Fig. 2. According to this convention, rotation of a crystal by an angle Θ about axis OY' results returning to a horizontal position of a crystal. In this case, the angle Ξ defining a rotation about a vertical axis is between a global axis OX and projection of an axis O'X' onto a plane XY.

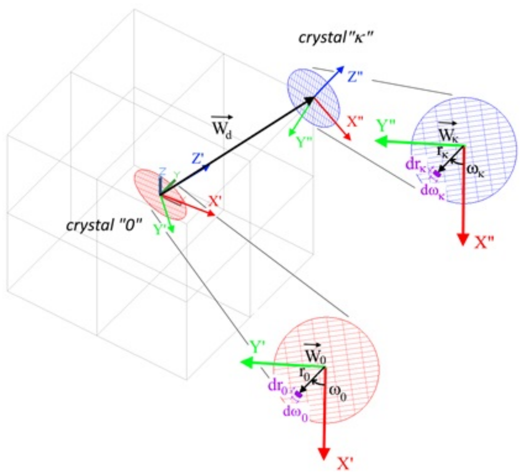

Figure 2: Local coordinate systems for each of the crystals.

The modelling is based on calculating the moments of electrostatic forces acting on the central crystal from adjacent crystals, cf. Fig. 1. This procedure requires a calculation of the quadruple integral:

$$\overrightarrow{M} = \sum_{k=1}^{N} 2 \int_0^{2\pi} \int_0^{2\pi} \int_0^{R} \int_0^{R} \overrightarrow{W_0} \times \overrightarrow{F} \mathrm{d}r_0 \mathrm{d}r_\kappa \mathrm{d}\omega_0 \mathrm{d}\omega_\kappa \tag{1}$$

where N is the number of acting crystals,

$\overrightarrow{F}(r_0, \omega_0, r_\kappa, \omega_\kappa)$ is the electrostatic force between the two infinitesimal fragments of crystal, cf. Fig. 2. According to the Coulomb's law this force is given by:

$$\overrightarrow{F}(r_0, \omega_0, r_\kappa, \omega_\kappa) = \frac{dq \cdot dq}{2\pi\varepsilon_\gamma\varepsilon_0} \cdot \frac{\overrightarrow{W_{0\kappa}}}{|\overrightarrow{W_{0\kappa}}|^3} \tag{2}$$

where:

dq is signed magnitudes of the electric charge of infinitesimal fragment of a crystal,

ε_0 is the vacuum permittivity,

ε_κ is the relative permittivity of an air.

Combined numerical calculation of the integral (1) and Taylor expansion we finally obtained series of functions which are equation coefficients. These coefficients are dependent of unknowns angles and determined for each k independent. This has provided a discrete description of the considered structure. Estimating the continuous data was possible by using finite difference method for three-dimensional grid. Formulas for the partial derivatives for the FDM grid coinciding with the distribution of crystals in the cloud (cf. Fig. 1) can be obtained by reusing of Taylor expansion and taking into account FDM properties. Finally, searched system of equations can be presented in the form:

$$\begin{cases} \frac{\partial^2\Xi}{\partial t^2} I_x = M_0^{x0} + M_{xx}^{x\Theta}\Theta_{,xx} + M_{yy}^{x\Theta}\Theta_{,yy} + M_{zz}^{x\Theta}\Theta_{,zz} + \\ \quad + M_{xy}^{x\Theta}\Theta_{,xy} + M_{xz}^{x\Theta}\Theta_{,xz} + M_{yz}^{x\Theta}\Theta_{,yz} + \\ \quad + M_{xx}^{x\Xi}\Xi_{,xx} + M_{yy}^{x\Xi}\Xi_{,yy} + M_{zz}^{x\Xi}\Xi_{,zz} + \\ \quad + M_{xy}^{x\Xi}\Xi_{,xy} + M_{xz}^{x\Xi}\Xi_{,xz} + M_{yz}^{x\Xi}\Xi_{,yz} \\ \frac{\partial^2\Theta}{\partial t^2} I_y = M_0^{y0} + M_{xx}^{y\Theta}\Theta_{,xx} + M_{yy}^{y\Theta}\Theta_{,yy} + M_{zz}^{y\Theta}\Theta_{,zz} + \\ \quad + M_{xy}^{y\Theta}\Theta_{,xy} + M_{xz}^{y\Theta}\Theta_{,xz} + M_{yz}^{y\Theta}\Theta_{,yz} + \\ \quad + M_{xx}^{y\Xi}\Xi_{,xx} + M_{yy}^{y\Xi}\Xi_{,yy} + M_{zz}^{y\Xi}\Xi_{,zz} + \\ \quad + M_{xy}^{y\Xi}\Xi_{,xy} + M_{xz}^{y\Xi}\Xi_{,xz} + M_{yz}^{y\Xi}\Xi_{,yz} \end{cases} \tag{3}$$

where I_x and I_y are moments of inertia about OX and OY axis, M_0^{x0}, $M_{xx}^{x\Theta}$, $M_{yy}^{x\Theta}$, $M_{zz}^{x\Theta}$ are the functional coefficients of angles Θ and Ξ which can be described by matrices of coefficients and the vectors containing the trigonometric functions.

4. Remarks

Obtained result (3) is the system two partial differential equations of second order with respect to the three spatial coordinates and time. Solutions derived from such system of equations may be used to theoretical analysis of the dynamic behaviour of real, atmospheric clouds of ice crystals and modelling an optical phenomenon called the "miracle of the sun". Finding a general analytical solution of the above system of equations is impossible and seeking the numerical solutions will be the subject of further authors research.

References

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