New trends in continuum mechanics and challenges for numerical mechanics

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Abstract

Continuum mechanics is a branch of mechanics that deals with the analysis of the mechanical behavior of materials modeled as a continuous manifold. Continuum mechanics models begin by introducing of three-dimensional (or less dimensional) Euclidean space. The points within this region are defined as material points with prescribed properties. Each material point is characterized by a position vector which is continuous in time. Thus, the body changes in a way which is realistic, globally invertible at all times and orientation-preserving, so that the body cannot intersect itself and as transformations which produce mirror reflections are not possible in nature. For the mathematical formulation of the model it is also assumed to be twice continuously differentiable, so that differential equations describing the motion may be formulated. Finally, the kinematical relations, the balance equations, the constitutive equations and the boundary and/or initial conditions has to defined.

Within the presentation some examples of *solid deformable continua* will be discussed regarding the basics and the numerical treatment which is in general non-continuous. Finally, advanced models of continuum mechanics will be introduced and the challenges for the numerical mechanics will be formulated.

Keywords: continuum mechanics, numerical methods, balance equations, constitutive equations

1. Continuum Mechanics

Modeling a solid or fluid as a continuum assumes that the substance of the object completely and continuously fills the volume it occupies. Modeling objects as continua ignores that matter is made of atoms, and so is not continuous. Fundamental physical laws such as the balance of mass, the balance of momentum, the balance of moment of momentum, the balance of energy and the balance of entropy may be applied to this model to derive integral or differential equations describing their behavior. Finally, some information about the particular material behaviour is added through constitutive and evolution equations.

Continuum mechanics deals with physical properties which are independent of any particular coordinate system in which they are observed. These physical properties are then represented by tensors. They are mathematical objects which can be presented as invariant quantities using the direct tensor calculus, see e.g. [1]. These tensors can be expressed in coordinate systems for computational convenience.

2. Basic Equations

The basic equations of continuum mechanics are partly independent from the material behaviour and can be formulated in a more or less general manner. The materials behavior is the individual response with respect to the loading. Below the basic statement are given similar to [2, 3]

2.1. Material-Independent Statements - Classical Continuum Mechanics

The starting point of continuum mechanics is the kinematics. Within the kinematics there are two elements: deformation and motion. Both can be presented mathematically by some geometrical considerations. The kinematics can be introduced without statements concerning the reasons of deformation and motion, but also assumptions about the material behavior.

In the simplest case two configurations are given: the reference configuration for initial time and the current configuration. Any change in the configuration of the continuum results in a displacement having two parts: a rigid-body motion and a deformation. A rigid-body motion (translation and rotation) is given when the body does not change its shape or size. The motion and the deformation of a continuum is a continuous time sequence. Thus, the material body will occupy different configurations at different times.

Any equation or property can be presented by Langrangian or Eulerian descriptions. In the Lagrangian description dominantly in solid mechanics the position and physical properties of the particles are described in terms of the material or reference configuration and time. As usual, the reference configuration is the configuration at $t = 0$. An observer standing in the referential frame observes the changes in the position and physical properties as the material body moves in space as time progresses. The Eulerian description focuses on the current configuration, giving attention to what is occurring at a fixed point in space as time progresses, instead of giving attention to individual particles as they move through space and time. This approach is conveniently applied in fluid mechanics where the kinematic property of greatest interest is the rate at which change is taking place.

Continuum mechanics deals with deformable bodies. Following the classical dynamics, the motion of a material body is produced by the action of externally applied forces and moments. One should distinguish surface and body actions. They result in stresses within the body, but in the classical continuum mechanics the couple stresses are ignored. In addition, the behavior of the continuum can be influenced by non-mechanical actions (thermal, electric, magnetic, etc.).

The last step in formulation the material-independent equations is the presentation of the balances. Let us introduce an additive quantity

$$Y(t) = \int_V \Psi(\boldsymbol{x}, t)\mathrm{d}V \tag{1}$$

V is the volume in the actual configuration, $\boldsymbol{x}$ is the position vector and t denotes the time. The changes of the $Y(t)$ are induced by volume and surface effects

$$\frac{\mathrm{D}}{\mathrm{D}t}Y(t) = \frac{\mathrm{D}}{\mathrm{D}t}\int\limits_V \Psi(\boldsymbol{x},t)\mathrm{d}V = \int\limits_A \Phi(\boldsymbol{x},t)\mathrm{d}A + \int\limits_V \Xi(\boldsymbol{x},t)\mathrm{d}V \tag{2}$$

Equations (1) and (2) are valid for tensors Ψ of arbitrary rank. Ψ is a scalar quantity in the case of mass, energy and entropy balance and a vector quantity in the case of the balance of momentum and moment of momentum. Φ is the flux through the surface A in the direction of the normal $\boldsymbol{n}$. Ξ denotes the increase in the volume V. It is obvious that Φ is a tensor one rank higher in comparison with Ψ and Ξ. Finally, following the standard procedures have to be performed to establish the local form of the balances [2]

- applying the Reynolds theorem the balance equation in the reference configuration,
- the transform of the surface integral into a volume integral applying the Gauß-Ostrogradski theorem and
- assuming the smoothness of all fields.

2.2. *Material-Dependent Statements*

There are two types of statement: the constitutive equations and may be some evolution equations. The first one is a relation between two or more physical quantities. Examples are stresses, strains, strain rates, heat flux, etc. These equations are specific to a material or substance, and approximates the response of that material to external mechanical or non-mechanical stimuli. They can be formulated top-down (from the general model to more simpler models), bottom-up (starting from the simplest one and performing step-wise extensions) and using rheological models. If there is an additional evolution process (for example damage) evolution equations can be introduced. Constitutive equations can be of the algebraic, differential, integral and other type. Evolution equations are ordinary differential equations of first order with respect to time.

There are a lot of suggestions for material behavior models. The most popular are the elastic, the visco-elastic, the plastic and visco-plastic constitutive equations, see [3] among others.

2.3. *Extensions of the Classical Continuum Mechanics*

There are several extensions suggested since the end of the nineteenth century. The first one was the model of the Cosserat brothers [4]. This extension is related to the introduction ones not only translational degrees of freedom, but also rotational ones. Later by Eringen and others different variants of the Cosserat approach were introduced, see for example [5].

Another possibility is given by using higher order gradients [6]. In the classical continuum mechanics we have only the first gradient of the displacements ($\boldsymbol{\nabla u}$). Introducing $\boldsymbol{\nabla\nabla u}$ or higher order gradients one can take into account not only the infinitesimal surrounding of the material points. Point forces and force distributions along lines are basic loading conditions in mechanics. Such boundary conditions along lines or on points result in singularities of the displacement field. If one wants to introduce point and line forces (or point and line displacements, respectively), one has to generalize the concept of the Cauchy continuum. An extension of the elastic energy of the continuum to second and third gradients of the displacement lends itself to this purpose. From the extensions of Mindlin [7] and Germain [8], that have been further developed in e.g. [9], it becomes obvious why the introduction of the first and second strain gradient allow a continuum to sustain boundary conditions on vertices and edges of a body. Another important feature of strain gradient theories is their regularization property. The strain gradient model in Aifantis [10] is discussed with respect to its regularization property. It is shown by Lazar and Maugin [11] that this model for small deformations still has singularities in the higher-order stresses. In a following work they show for small deformations that for dislocation problems in an infinite plane a second strain gradient theory produces no singularities at all [12]. This result is confirmed with applications in dislocation analysis [13].

3. Numerical Treatment

Many materials, although showing considerable complexity in their structure and interior architecture, can be modeled at a small length scale by a classical Cauchy medium. Such a model can be characterised by a very large number of degrees of freedom even for a small sample. This choice, on the one hand, allows us to use standard numerical tools based on finite element methods, which are optimized for this kind of model. On the other hand, the complexity of the considered continua makes the use of such a model unsuitable from the point of view of computational costs. Gradient models allow to obtain sufficiently accurate solutions (see e.g. [14, 15]), comparable with those of the Cauchy theory, but at much smaller computational costs. There are many examples for materials where the corresponding continuum models are obtained by a homogenisation procedure which leads to micromorphic generalized continua. Gradient materials are a very particular case of such micromorphic continua (see e.g. [16, 17, 18, 19] for generalized continua with microstructure and [20, 21, 22] for higher-gradient continua). Gradient models involve an increase of material parameters necessary to describe the more detailed characterization of deformations. Therefore, specific tests, both experimental and numerical, can be designed to identify such parameters as proposed in [23] or in [24, 25] for dynamic properties as dissipation coefficients or parameters related to frequency response functions. With regards to the numerical method to solve the governing equations, one of the recent developments of the finite element method (FEM), Isogeometric Analysis (IGA), can be used. Introduced with the aim of filling the gap between computer-aided geometric design (CAGD) and analysis worlds, IGA has attracted the interest of many researchers recently. The concept was introduced by the group of Hughes in 2005, where they proposed to use NURBS (Non-Uniform Rational Basis Splines) as the basis function both for the geometry and the analysis [26]. NURBS are the industry standard in CAGD, and using them as the basis of the analysis will eliminate the need to approximate the geometry to have an analysis suitable model, hence more accurate results. Besides, using NURBS shape functions will provide a smoother solution field (higher continuity). In 2011, they proposed a new data structure for IGA based on Bezier extraction of NURBS [27]. They showed that using their method, it is possible to implement the IGA concept in already-developed finite element codes at the cost of calculating an extraction operator. In their method, numerical integration of smooth functions will be carried out on C_0 Bezier elements using the extraction operator. It also makes it possible to use other spline technologies (such as T-Splines) as the analysis basis without much difficulty. Numerical solution of the gradient elasticity equations needs higher continuities of the solution field. This is one of the issues where the IGA method can show its advantages. In 2011, Fischer et al. [28] used IGA to solve the problem of gradient elasticity in two dimensions and exploited the higher continuity nature of the NURBS basis functions to overcome the need of introducing auxiliary degrees of freedom. Since then, many studies have been performed based on the isogeometric analysis and its application in the gradient elasticity theory; namely, [29, 30, 31] among others.

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