Equilibrium finite element method for Kirchhoff's plate

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Abstract

An alternative formulation of the thin plate equilibrium problem compared to the most popular displacement approach is considered in the paper. A stress-based formulation of the finite element method is applied. The statically admissible field of section moments is constructed by means of Southwell's stress function. The solution of the problem is found by minimization of the functional of complementary energy. The numerical results obtained by the proposed method are compared to the approximate solution given by the displacement approach.

Keywords: equilibrium model, finite element method, Kirchhoff's plate model, Southwell's stress function

1. Introduction

The equilibrium approach to the finite element (FE) method is rarely used in research and engineering practice compared to the displacement, mixed or hybrid methods. The reason of this fact follows from some difficulties in construction of statically admissible stress fields. Nevertheless, the equilibrium approach has some attractive features like the possibility to determine the lower bounds for rigidity and load limit of analysed systems or easy estimation of the error of the approximate solution when this approach is coupled with the displacement method.

This work presents the application of the equilibrium finite element method to the static problem of Kirchhoff's plate bending. The statically admissible fields of stresses have been constructed with use of the Southwell's stress function. The possibility of employing this function for construction of the approximate solution for analysis of plates was indicated by Zienkiewicz and Fraeijs de Veubeke in [1]. Southwell's functions components are approximated by means of triangular elements of class C^0 with parabolic shape functions. The solution has been obtained by minimizing the functional of complementary energy. Equilibrium conditions on the plate boundary have been satisfied by use of the method of Lagrange's multiplier.

2. Problem formulation

Let Ω denote the region representing the plate mid-surface area, and $\partial\Omega$ its boundary. In the case of the linear thin plate theory, the following equilibrium equations, Eq. (1), and curvature–deflection relation, Eq. (2), are satisfied on Ω:

$$M_{\alpha\beta,\beta} - Q_\alpha = 0, \quad Q_{\alpha,\alpha} + q = 0, \tag{1}$$

$$\kappa_{\alpha\beta} = -w_{,\alpha\beta} \tag{2}$$

where $\alpha, \beta = 1, 2$, and $M_{\alpha\beta}$ denotes the tensor of bending and twisting moments, Q_α is the vector of transverse forces, q the transverse surface load while $\kappa_{\alpha\beta}$ and w denote the tensor of curvature and deflection, respectively. The linear constitutive relation takes the following form:

$$\kappa_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\, M_{\gamma\delta} \tag{3}$$

where $C_{\alpha\beta\gamma\delta}$ indicates the compliance tensor which, for an isotropic material, can be expressed as follows: $C_{\alpha\beta\gamma\delta} = 12\,[(1+\nu)\,\delta_{\alpha\gamma}\,\delta_{\beta\delta} - \nu\,\delta_{\alpha\beta}\,\delta_{\gamma\delta}]/(E\,h^3)$, where E is Young's modulus, ν Poisson's ratio, h thickness of the plate and $\delta_{\alpha\beta}$ Kronecker's symbol. To complete the considered boundary value problem, the essential boundary conditions are formulated

$$w = 0, \quad w_{,n} = 0 \quad \text{on } \Gamma_w \subset \partial\Omega \tag{4}$$

where $w_{,n}$ denotes the derivative normal to the boundary, and the natural boundary conditions

$$M_n = M_0, \quad \bar{Q} \equiv Q_n + M_{ns,s} = Q_0 \quad \text{on } \Gamma_\sigma \subset \partial\Omega \tag{5}$$

where indicies n ans s are related to local axes normal and tangential to boundary $\partial\Omega$, respectively, and $\bar{Q}$ denotes the effective shear force while M_0 and Q_0 are given functions of variable s. It is assumed in Eqs. (4) and (5) that $\Gamma_w \cap \Gamma_\sigma = \emptyset$ and $\bar{\Gamma}_w \cup \bar{\Gamma}_\sigma = \partial\Omega$.

The variational formulation has been utilised to set the finite element equations. Let the following set of statically admissible fields of section moments be defined:

$$Y = \big\{ M_{\alpha\beta} \in L^2(\Omega) : M_{\alpha\beta,\alpha\beta} = -q \text{ in } \Omega, \\ M_n = M_0 \;\&\; \bar{Q} = Q_0 \text{ on } \Gamma_\sigma \big\}\,. \tag{6}$$

Multiplication of Eq. (2) by difference $(T_{\alpha\beta} - M_{\alpha\beta})$, integration over region Ω, then use of Green's formula and the constitutive equation leads to the weak formulation which has a form of the principle of complementary work: Find $M_{\alpha\beta} \in Y$ such that the following equation is satisfied:

$$\int_\Omega M_{\alpha\beta}\, C_{\alpha\beta\gamma\delta}\,(T_{\gamma\delta} - M_{\gamma\delta})\,\mathrm{d}x = 0 \quad \forall\, T_{\alpha\beta} \in Y. \tag{7}$$

Formulation (7) is equivalent to the problem of minimisation of the functional of complementary energy on set Y

$$\Sigma(\boldsymbol{M}) = \frac{1}{2} \int_\Omega M_{\alpha\beta}\, C_{\alpha\beta\gamma\delta}\, M_{\gamma\delta}\,\mathrm{d}x.$$

3. Solution to the problem

The equilibrium equations have been satisfied at internal points of region Ω by means of Southwell's vector stress function [2, 1] with its components denoted by U and V. The equilibrium equations (1) are satisfied when vector $\boldsymbol{\sigma}$ of bending and twisting moments is expressed as follows:

$$\boldsymbol{\sigma} \equiv \begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial V}{\partial x_2} \\ \frac{\partial U}{\partial x_1} \\ -\frac{1}{2}\left(\frac{\partial U}{\partial x_2} + \frac{\partial V}{\partial x_1}\right) \end{bmatrix} - \begin{bmatrix} P_0 \\ P_0 \\ 0 \end{bmatrix} \tag{8}$$

where P_0 is a solution of Poisson's equation, $\nabla^2 P_0 = q$. As the generalized stress vector, $\boldsymbol{\sigma}$, depends on first order derivatives of Southwell's function, it may be approximated by use of finite elements with shape functions of class C^0. This is in contrast to the displacement approach to FE method where the continuity of derivatives of deflection function in the direction normal to the edge of the element is necessary to guarantee convergence of the approximate solution. The stress function has been expressed by means of the matrix of shape functions $\mathbf{N}$ and the vector of degrees of freedom $\mathbf{a}$ as follows:

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots \\ 0 & N_1 & 0 & N_2 & \dots \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix} \equiv \mathbf{N}\,\mathbf{a} \tag{9}$$

where the number of diagonal matrixes of size 2×2 equals to the number of nodes of the FE mesh. Substitution of Eq. (9) to Eq. (8) leads to the following form of the generalized stress vector:

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \frac{\partial N_1}{\partial x_2} & 0 & \dots \\ \frac{\partial N_1}{\partial x_1} & 0 & \frac{\partial N_2}{\partial x_1} & \dots \\ -\frac{1}{2}\frac{\partial N_1}{\partial x_2} & -\frac{1}{2}\frac{\partial N_1}{\partial x_1} & -\frac{1}{2}\frac{\partial N_2}{\partial x_2} & \dots \end{bmatrix} \mathbf{a} - \mathbf{P}_0 \equiv$$
$$\equiv \mathbf{B}\,\mathbf{a} - \mathbf{P}_0 \tag{10}$$

where $\mathbf{P}_0 = [P_0\ P_0\ 0]^{\mathrm{T}}$. By virtue of Eq. (10), the equation of complementary work (7) may be written in the matrix form:

$$\delta\mathbf{a}^{\mathrm{T}} \int_{\Omega} \mathbf{B}^{\mathrm{T}}\,\mathbf{C}\,(\mathbf{B}\,\mathbf{a} - \mathbf{P}_0)\,\mathrm{d}x = 0, \tag{11}$$

which is satisfied for any variation of vector of degrees of freedom $\delta\mathbf{a}$ with constraints that follow from the boundary conditions. In the equation above, $\mathbf{C}$ denotes the plate compliance matrix of size 3×3. Eq. (11) is equivalent to the following set of linear algebraic equations:

$$\mathbf{K}\,\mathbf{a} = \mathbf{F} \quad \text{with} \tag{12}$$
$$\mathbf{K} = \int_{\Omega} \mathbf{B}^{\mathrm{T}}\,\mathbf{C}\,\mathbf{B}\,\mathrm{d}x, \quad \mathbf{F} = \int_{\Omega} \mathbf{B}^{\mathrm{T}}\,\mathbf{C}\,\mathbf{P}_0\,\mathrm{d}x.$$

The traction boundary conditions (5) that occur in the definition of set of statically admissible fields of moments Y (6) have been satisfied with the use of Lagrange's multiplier method.

4. Numerical example

A trapezoidal uniformly loaded plate ($q = 5\,\mathrm{kN/m^2}$) with bases of length 4 and 2 m, and a side perpendicular to them of 3 m in length has been analysed. The shorter base and inclined edge have been assumed to be clamped while two other sides simply supported. Computations have been made with the following material data: Young's modulus, $E = 30\,\mathrm{GPa}$, Poisson's ratio, $\nu = 0.2$, and the plate thickness, $h = 0.2\,\mathrm{m}$.

The convergence of the proposed statically admissible solution has been checked using several computational grids with various element sizes: 0.1, 0.2, 0.4 and 0.8 m. The results have been compared to those obtained by use of the displacement-based FE method with the help of the Hsieh–Clough–Tocher triangular macroelement. The contour plots for bending moments obtained with the mesh size 0.2 m by the present stress-based FE model are shown in Fig. 1. The displacement based solution has exhibited slight differences and is not depicted here.

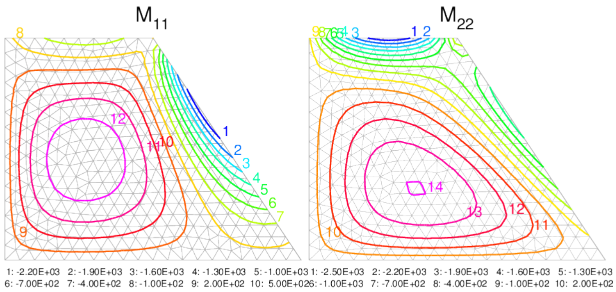

Figure 1: Bending moments [N], mesh size: 0.2 m

The values of strain energy and relative error of the approximate solution in relation to the element size of the mesh are drawn in Fig. 2. The combined use of two dual approaches to the FE method allows one to find upper and lower bounds for the strain energy and estimate *a posteriori* the error of the approximate solution. This error has been calculated using the Synge hipercircle method and appeared to be smaller than 0.5% in the case of the densest mesh.

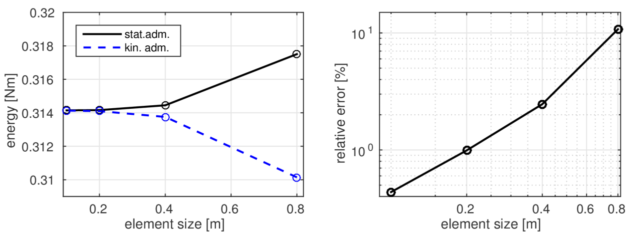

Figure 2: Strain energy and relative energy error of the solution

5. Conclusions

A triangular element of C^0 class with parabolic interpolation functions satisfying the equilibrium conditions has been applied to the thin plate bending problem. The two-component Southwell stress function has been utilised. The obtained statically admissible solution has been compared to the kinematically admissible one; very good agreement between both the approximate solutions has been obtained. The error of the approximate solution has been estimated. Very small value of the error has been determined – quite satisfactory from the engineering viewpoint.

References

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