On material design by the Pareto optimal choice of elastic moduli distribution

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Abstract

The present paper concerns the problem of Pareto optimal distribution of isotropic material properties: bulk and shear moduli or Young modulus with prescribed, not necessarily uniform, distribution of Poisson's ratio in the Isotropic Material Design (IMD) and Young Modulus Design (YMD) method, respectively. The minimized objective function is the weighted sum of compliances corresponding to n independent loading conditions. The only isoparametric condition is the integral of the unit cost of the design assumed to be equal to the trace of the elastic moduli tensor **C**. Both the methods: IMD and YMD, reduce to the auxiliary minimization problem involving statically admissible stress fields corresponding to the subsequent loading conditions.

Keywords: vector optimization, compliance minimization, isotropic material design

1. Formulation of the IMD/YMD vector optimization

The paper refers to the current research in the free material design, cf. Refs [1-3]. The isotropic material design (IMD) proposed in Ref. [2] and the Young modulus design delivered recently in Ref. [3] are extended to the multiple load cases.

Consider the problem of vector minimization of the compliances of a structure subjected, non-simultaneously, to $n\geq 1$ traction loads $\mathbf{t}^\alpha$, $\alpha=1,2,\ldots,n$ acted on the given part Γ_1 of the boundary $\partial\Omega$ of the given spatial design domain Ω. Let us define 4-th rank tensors

$$\mathbf{S}_1=\frac{1}{3}\delta_{ij}\delta_{kl}\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_l\,,\ \mathbf{S}_2=\mathbf{II}-\mathbf{S}_1\,, \tag{1}$$

where $\mathbf{e}_i\in\mathrm{R}^3$, i = 1,2,3 is orthogonal basis in three dimensional Euclidean space parameterized by the Cartesian system $x=(x_1, x_2, x_3)$ and $\mathbf{II}=(1/2)(\delta_{ik}\,\delta_{jl}+\delta_{il}\,\delta_{jk})\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_l$ - the identity tensor in the space of symmetric, 4-th rank tensors E_s^4. The design variables are: the bulk $k=k(x)$ and the shear moduli $\mu=\mu(x)$, or: Young's modulus $E=E(x)$ (for prescribed in Ω field of Poisson ratio $\nu=\nu(x)$) defining the non-homogeneous, isotropic, 4-th rank Hooke tensors **C**:

$$\mathbf{C}(x)=3k(x)\mathbf{S}_1+2\mu(x)\mathbf{S}_2\,,$$
$$\mathbf{C}(x)=\frac{E(x)}{1-2\nu(x)}\mathbf{S}_1+\frac{E(x)}{1+\nu(x)}\mathbf{S}_2 \tag{2}$$

$x\in\Omega$, for IMD and YMD method, respectively. Let us define the linear form

$$\forall\mathbf{v}\in V(\Omega)\quad f^\alpha(\mathbf{v})=\int_{\Gamma_1}\mathbf{t}^\alpha\cdot\mathbf{v}\,ds\,, \tag{3}$$

where $\mathbf{v}=(v_1,v_2,v_3)$ is a virtual, kinematically admissible field satisfying the condition $\mathbf{v}=\mathbf{0}$ on the given boundary (support) $\Gamma_2\subset\partial\Omega$, $\Gamma_1\cap\Gamma_2=\varnothing$. The compliance corresponding to the α-th load is defined by

$$\wp^\alpha(\mathbf{C})=f^\alpha(\mathbf{u}^\alpha)\,, \tag{4}$$

where $\mathbf{u}^\alpha=(u_1^\alpha, u_2^\alpha, u_3^\alpha)$ is the displacement field for the α-th load.

The formulation of the IMD, or YMD problem: find Hooke tensor **C** minimizing all compliances $\wp^\alpha=\wp^\alpha(\mathbf{C})$, $\alpha=1,2,\ldots,n$ at the same time and satisfying the cost (isoparametric) condition

$$\int_\Omega \mathrm{tr}\,\mathbf{C}\,dx=\Lambda,\quad \Lambda=E_0|\Omega|=const\,, \tag{5}$$

where E_0 is the referential elastic modulus, is mathematically incorrect. The requirement of minimizing all objectives $\wp^\alpha$ simultaneously may be replaced, see Ref. [4], by the requirement of finding all Pareto optimal Hooke tensors $\mathbf{C}^*$. One of the most common method of finding Pareto optimal solutions is the weighted sum approach that transforms the vector optimization problem into scalar valued optimization problem: find feasible Hooke tensor $\mathbf{C}^*$ minimizing convex combination of the compliances

$$\wp_{\mathbf{q}}(\mathbf{C})=\sum_{\alpha=1}^{n}\eta_\alpha\wp^\alpha(\mathbf{C}),\ \sum_{\alpha=1}^{n}\eta_\alpha=1,\ \eta_\alpha\geq 0,\ \alpha=1,\ldots,n \tag{6}$$

corresponding to subsequent load conditions $\mathbf{t}^\alpha$, the importance of which is reflected in weighting factors η_α forming the vector of weights $\eta=(\eta_1,\eta_2,\ldots,\eta_n)\in\mathrm{R}^n$. Let $\Sigma_\alpha(\Omega)$ be the set of admissible stresses $\mathbf{U}=(\tau_{ij})$ corresponding to the traction load $\mathbf{t}^\alpha$. By inverting (2) and expressing the compliance (4) by stress fields, according to Castiglianoôs theorem it is possible to reformulate the problem (6) for arbitrary but fixed vector of weights $\eta=(\eta_1,\eta_2,\ldots,\eta_n)\in\mathrm{R}^n$ as follows: find two fields $k_{\mathbf{q}}^*,\mu_{\mathbf{q}}^*$, or field $E_{\mathbf{q}}^*$, fulfilling (5) such that

$$\wp_{\mathbf{q}}^*=\wp_{\mathbf{q}}(k_{\mathbf{q}}^*,\mu_{\mathbf{q}}^*)=\min_{k\geq 0,\mu\geq 0}\wp_{\mathbf{q}}(k,\mu) \tag{7}$$

or

$$\wp_{\mathbf{q}}^*=\wp_{\mathbf{q}}(E_{\mathbf{q}}^*)=\min_{E\geq 0}\wp_{\mathbf{q}}(E) \tag{8}$$

in IMD or YMD method, respectively. Let us define the two norms, denoted by the same symbol $|\cdot|$:

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$$\left|\left(\mathring{\mathbf{u}}^1,\ldots,\mathring{\mathbf{u}}^n\right)\right| = \frac{\sqrt{3}}{3}\sqrt{\sum_{\alpha=1}^{n}\left(\operatorname{tr}\mathring{\mathbf{u}}^\alpha\right)^2} + \sqrt{5}\sqrt{\sum_{\alpha=1}^{n}\left\|\operatorname{dev}\mathring{\mathbf{u}}^\alpha\right\|^2}\ ,$$

$$\left|\left(\mathring{\mathbf{u}}^1,\ldots,\mathring{\mathbf{u}}^n\right)\right| = \sqrt{\frac{6-9\nu}{3(1+\nu)}\sum_{\alpha=1}^{n}\left(\operatorname{tr}\mathring{\mathbf{u}}^\alpha\right)^2 + \frac{6-9\nu}{1-2\nu}\sum_{\alpha=1}^{n}\left\|\operatorname{dev}\mathring{\mathbf{u}}^\alpha\right\|^2} \tag{9}$$

for IMD and YMD cases, respectively. In the above formulae $\mathring{\mathbf{u}}^\alpha$ being the second rank symmetric tensors. The most important result is the following lemma:

let $\mathbf{q} \in R^n$ be arbitrary but fixed vector of weights. Then the non-negative fields:

$$k^*_{\mathbf{q}} = \frac{\Lambda}{3\sqrt{3}}\frac{\sqrt{\sum_{\alpha=1}^{n}\left(\operatorname{tr}\left(\sqrt{\eta_\alpha}\,\mathring{\mathbf{U}}^\alpha\right)\right)^2}}{\int_\Omega\left|\left(\sqrt{\eta_1}\,\mathring{\mathbf{U}}^1,\ldots,\sqrt{\eta_n}\,\mathring{\mathbf{U}}^n\right)\right|dx}\ , \tag{10}$$

$$\mu^*_{\mathbf{q}} = \frac{\sqrt{5}}{10}\Lambda\frac{\sqrt{\sum_{\alpha=1}^{n}\left\|\operatorname{dev}\left(\sqrt{\eta_\alpha}\,\mathring{\mathbf{U}}^\alpha\right)\right\|^2}}{\int_\Omega\left|\left(\sqrt{\eta_1}\,\mathring{\mathbf{U}}^1,\ldots,\sqrt{\eta_n}\,\mathring{\mathbf{U}}^n\right)\right|dx} \tag{11}$$

are solutions of the IMD problem (7) and the non-negative field

$$E^*_{\mathbf{q}} = \frac{(1+\nu)(1-2\nu)\Lambda}{6-9\nu}\frac{\left|\left(\sqrt{\eta_1}\,\mathring{\mathbf{U}}^1,\ldots,\sqrt{\eta_n}\,\mathring{\mathbf{U}}^n\right)\right|}{\int_\Omega\left|\left(\sqrt{\eta_1}\,\mathring{\mathbf{U}}^1,\ldots,\sqrt{\eta_n}\,\mathring{\mathbf{U}}^n\right)\right|dx} \tag{12}$$

is the solution of the YMD problem (8), where

$$\left(\mathring{\mathbf{U}}^1,\ldots,\mathring{\mathbf{U}}^n\right) \in \Sigma_1(\Omega)\times\ldots\times\Sigma_n(\Omega) \tag{13}$$

is the solution of the following problem

$$\int_\Omega\left|\left(\sqrt{\eta_1}\,\mathring{\mathbf{U}}^1,\ldots,\sqrt{\eta_n}\,\mathring{\mathbf{U}}^n\right)\right|dx = \min_{\left(\mathring{\mathbf{U}}^1,\ldots,\mathring{\mathbf{U}}^n\right)\in\Sigma_1(\Omega)\times\ldots\times\Sigma_n(\Omega)}\int_\Omega\left|\left(\sqrt{\eta_1}\,\mathring{\mathbf{U}}^1,\ldots,\sqrt{\eta_n}\,\mathring{\mathbf{U}}^n\right)\right|dx. \tag{14}$$

with the norm involved defined by $(9)_1$ or $(9)_2$ for IMD and YMD, respectively.

2. Example: Pareto-optimal cantilever and final remarks

The IMD and YMD problems $(9)_{1,2}$ are solved numerically by using an interpolation of statically admissible stress fields, see e.g. Refs [2,3].

The example concerns the three-dimensional prismatic, cantilever rod $L_1 \times L_2 \times L_3$ = 0.7×0.7×2.0 [m], clamped along its bottom facet, see Fig.1. The upper horizontal facet is subjected to two independent, centrally applied forces $\mathbf{t}^1 = \left(t_1^1, 0, 0\right)^T$, $\mathbf{t}^2 = \left(0, t_2^2, 0\right)^T$ modeled by the weight functions in such a way that the resultant surface integrals $\int_A t_i^\alpha da$, (i, α = 1, 2), $A = \left\{x \in \partial\Omega \middle| x_3 = L_3\right\}$ are equal to 1.0 [N] and are causing independent lateral bending deformations in x_1x_3 and x_2x_3 planes (cases I and II). Calculations were carried out by IMD and YMD methods for two variants of vector of weights: $\mathbf{q} = (0.5\,,0.5)$ and $\mathbf{q} = (0.9\,,0.1)$, see Fig.2. The unity value of elastic modulus in (5) is assumed, i.e. E_0=1.0 [N/m^2]. The final optimal layouts reflect the symmetry and asymmetry of the weight factors applied.

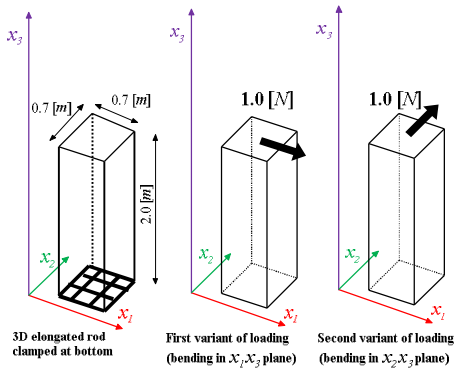

Figure 1: Three-dimensional slender body, clamped at the bottom, subjected to two independent loadings

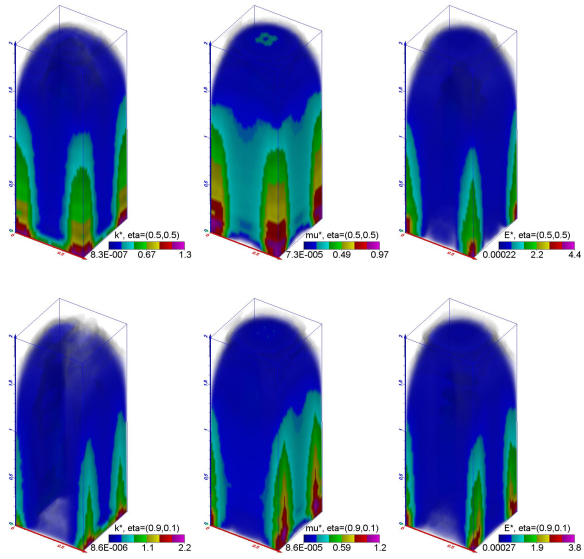

Figure 2: From top to bottom: first: $\mathbf{q} = (0.5\,,0.5)$ and second: $\mathbf{q} = (0.9\,,0.1)$ variant of loading. From left to right: Pareto-optimal distribution of bulk $k^*_{\mathbf{q}}$ and shear $\mu^*_{\mathbf{q}}$ modulus (IMD version) and Youngôs modulus $E^*_{\mathbf{q}}$ for $\nu = 0.3$ (YMD version).

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