On the error analysis of the Meshless FDM and its Multipoint extension

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Abstract

The error analysis for the meshless methods, especially for the Meshless Finite Difference Method (MFDM) is discussed in the paper. Both a priori and a posteriori error estimations are considered. Experimental order of convergence confirms the theoretically developed a priori error bound. The higher order extension of the MFDM – the Multipoint approach may be used as a source of the improved reference solution instead of the true analytical one for the global and local error estimation of the solution and residual errors. Several types of a posteriori error estimators are described. A variety of performed tests confirm high quality of a posteriori error estimation based on the multipoint MFDM.

Keywords: error analysis, a priori error, a posteriori error, meshless finite difference method, multipoint meshless method

1. Introduction

In the field of engineering computation for the solution of practical engineering or physics problems, e.g. fracture mechanics, elasticity, structural and fluid dynamics, etc. the problem of overall accuracy of the computational modeling is determined by different types of errors. Among them there are:

- the inevitable error (error arising from the inadequacy of the mathematical model to real object, phenomenon or process; error noise in the input data);
- mathematical modeling error (disregarding of certain phenomena; simplification of data);
- numerical modeling error (error of a solution method, discretization, approximation, errors due to series truncation);
- computing error (round off error; truncation error; algorithmic error).

This paper concerns the numerical solution error, i.e. the difference between the exact and approximate solution. The exact solution usually is not known, therefore one can get only some estimates of the error. In numerical computations the error estimate is used to the *reliable assessment of the accuracy* of the computed solution and to the improvement of computational efficiency. In the second case the error estimation is used as the basis of an *adaptive refinement* process.

Both *a priori* and *a posteriori* errors may be considered, though the latter may be evaluated only after a solution of the problem is found. Instead, a priori error estimate is usually found before the whole solution process starts and yield information on the asymptotic behavior of the discretization errors.

The Finite Element Method (FEM) has been the most commonly applied method of computer analysis for many years. However, in many cases, e.g. the mesh distortion, frequent time-consuming remeshing is needed during the process of computation. This is the motivation of development of meshless methods which use only a set of nodes for discretization of the continuum. Nodes can be easily moved, inserted and deleted, therefore this technique leads to greater flexibility and is more convenient and attractive to implement adaptive process.

Many different meshless methods are proposed. Among other the Meshless Finite Difference Method (MFDM) [3] is one of the oldest as well as effective meshless method which is still under development.

Large amount of error analysis research have been developed and many error estimators are proposed in FEM [1, 2]. However, in the present paper, attention will be laid upon the error analysis for meshless methods. The developments of error estimation in the meshless methods will be shortly reviewed at the beginning. Then *a priopri* error estimation for the MFDM which was developed by using approach based on the moving least squares error bound [8] will be discussed. It estimates the solution convergence rate by using only mesh size h and approximation order p. Theoretically expected convergence can be proved by calculating so-called the *experimental order of convergence*. Next some types of the *a posteriori error* estimators based on the higher order extension of the MFDM – the Multipoint method used as the reference solution, are quoted in detail. The e*ffectivity index* has been introduced to measure the quality of a posteriori error estimators. To examine the proposed estimators several tests have been solved.

2. On the Meshless FDM and multipoint method

Meshless FDM [3], as other meshless methods, is based on the set of arbitrarily distributed nodes which may be locally remodeled without any difficulties. Additionally, the MFDM may use various formulations of the boundary value problems: strong (local), weak (global), and mixed (local-global) one. Moreover, it includes extensions allowing for higher order approximation, e.g. the multipoint MFDM [6], and what follows, it constitutes an effective tool for function approximation and error analysis.

In general, meshless approximation $u^h(x)$ for unknown function u may be written as

$$u(x) \approx u^h(x) = \sum_i \Phi_i(x) u_i \qquad (1)$$

where u_i are the nodal values (degrees of freedom, d.o.f.) of the unknown function $u(x)$ in the domain Ω and Φ_i are the shape (or pseudo-shape) functions, which may be generated using various methods of local approximation. In the case of MFDM, the shape functions are obtained by the Moving Weighted Least Squares (MWLS), [3, 7] approximation.

2.1. MWLS approximation

The idea of the MWLS is simple. To obtain the approximate value of unknown function u at point P_i the local or global approach may be used. In the case of local one some small selected subset of nodes P_j called the stencil is used (Fig.1). Construction of the local approximation of function is performed by assuming appropriate d.o.f. at nodes of stencil, e.g. the unknown function values. In the case of the multipoint MFDM – the special scheme with taking into account some additional d.o.f. [4] is used to obtain the higher order approximation. The basis functions of MWLS approximation may include the subsequent polynomials [7], or Taylor series expansion [3].

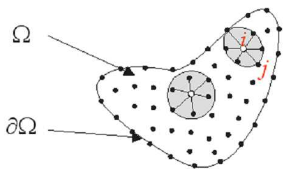

Figure 1: Selected stencils in domain Ω [4]

3. A priori error estimation

Evaluating the error estimates for the moving least squares (MLS) approximation in general case is one of the subjects of papers [8, 9]. This results in the local a priori error bounded by the error of a local polynomial approximation.

In the MWLS approximation used for the MFDM the Taylor polynomial of total degree p is considered instead of the subsequent polynomials as the basis function [3, 6]. In this research a priori error bound based on the investigation of the MLS approximation [8] is developed for the MFDM. We introduce an error formula which shows that MWLS approximation with Taylor polynomial reproduction of degree p has the approximation order $O(h^{p+1})$.

The resulting a priori bound may be confirmed by calculating the experimental order of convergence for the error, for instance, for two different meshsizes h_1 and h_2 as follows [4]

$$O_e(h) = \frac{\log(error_1 / error_2)}{\log(h_1 / h_2)} \ . \tag{2}$$

Its value is equal to k, if $error \le Ch^k$ with some constant $C > 0$.

A priori error estimate might be very effective, if it is applied to regular meshes and to simple linear differential operators. The convergence rate obtained by experimental order of convergence confirmed a priori error bound (Table 1.)

Table 1: A priori error bound $O(h^{p+1})$ examined by $O_e(h)$

Taylor pol. order p	O_e, $\sin(x+y)$ test	O_e, x^4+y^4+xy test	$p+1$
2	3,106	3,065	3
3	3,840	4	4
4	4,725	0, exact solution	5

4. A posteriori error estimation based on the multipoint MFDM

The multipoint approach provides higher accuracy solution that may be used as a reference solution in the local or global error estimation [5]. It may be applied for two purposes: to examine the accuracy of the computed solution and to generate series of adaptive meshes.

Several types of error estimators are considered:

Hierarchic estimators are based on the comparison of numerical results with the reference solution obtained with *h*-, *p*-, or *hp*- approach. The higher order multipoint method may be successfully applied as *p*- or *hp*- approach to compute improved solution. The error estimation quality may be measured by means of the so-called effectivity index I_{eff} – the ratio of some norm of the error estimate and the true error norm. I_{eff} = 1 for the true solution.

Residual estimators use either explicit residual errors or equivalent implicit ones. Each of them provides a quality measure of the higher (multipoint approach) or lower (standard MFDM) order solution error. Using the approximate higher order solution defined at the nodes, one may calculate the residuum between the nodes. Several tests done showed that the error distribution essentially depends on the smoothing parameter used in the MWLS approximation weight function

Zienkiewicz-Zhu smoothing error estimators are based on the comparison between the recovered (rough) and the reference (smoothed, e.g. higher order multipoint) derivatives [10].

It is worth mentioning here that in the MFDM as well as in the multipoint approach the unknown function derivatives are obtained without any additional computational cost. The best feature of MFD approaches is the same order of the convergence rate for derivatives as for the calculated solution.

5. Final remarks

The paper is focused on the error analysis in the Meshless FDM and its higher order multipoint extension. A priori bound and a posteriori error estimates of the solution and residual errors were developed. Further research in the field of the MFDM and multipoint method error analysis is planned.

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