Multimaterial Topology Optimization of Contact Problems using Phase Field Regularization

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Abstract

The paper is concerned with the analysis and the numerical solution of the multimaterial topology optimization problems for bodies in unilateral contact. The contact phenomenon with Tresca friction is governed by the elliptic variational inequality. The structural optimization problem consists in finding such topology of the domain occupied by the body that the normal contact stress along the boundary of the body is minimized. The original cost functional is regularized using the multiphase volume constrained Ginzburg-Landau energy functional. The first order necessary optimality condition is formulated. The optimal topology is obtained as the steady state of the phase transition governed by the generalized Allen-Cahn equation. The optimization problem is solved numerically using the operator splitting approach combined with the projection gradient method. Numerical examples are provided and discussed.

Keywords: topology optimization, unilateral contact problems, phase field regularization, operator splitting method

1. Introduction

Multimaterial topology optimization aims to find the optimal distribution of several elastic materials in a given design domain to minimize a criterion describing the mechanical or the thermal properties of the structure or its cost under constraints imposed on the volume or the mass of the structure [1]. In recent years multiple phases topology optimization problems have become subject of the growing interest [1, 4]. In contrast to single material design the use of multiple number of materials extends the design space and may lead to better design solutions. The paper is concerned with the structural topology optimization of systems governed by the variational inequalities. The class of such systems includes among others unilateral contact phenomenon [3] between the surfaces of the elastic bodies. This optimization problem consists in finding such topology of the domain occupied by the body that the normal contact stress along the boundary of the body is minimized. In literature [3] ,r4this problem usually is considered as two-phase material optimization problem with voids treated as one of the materials. In the paper the domain occupied by the body is assumed to consist from several elastic materials rather than two materials. Material fraction function is a variable subject to optimization. A necessary optimality condition for the topology optimization problem is formulated. The cost functional derivative is also used to formulate a gradient flow equation for this functional in the form of the generalized Allen–Cahn equation governing the evolution of the material phases. Two step operator splitting approach is used to solve this gradient flow equation. The optimal topology is obtained as a steady state solution to this equation. Finite difference and finite element methods are used as the approximation methods. Numerical examples are reported and discussed.

2. Problem Formulation

Consider deformations of an elastic body occupying two–dimensional bounded domain Ω with the smooth boundary Γ (see Fig. 1). The body is subject to body forces $f(x) = (f_1(x), f_2(x))$, $x \in \Omega$. Moreover, the surface tractions $p(x) = (p_1(x), p_2(x))$, $x \in \Gamma$, are applied to a portion Γ_1 of the boundary Γ. The body is clamped along the portion Γ_0 of the boundary Γ and the contact conditions are prescribed on the portion Γ_2. Parts Γ_0, Γ_1, Γ_2 of the boundary Γ satisfy: $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$, $i, j = 0, 1, 2, \Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$. The domain Ω is assumed to be occupied by $s \geq 2$ distinct isotropic elastic materials. Each material is characterized by Young modulus. The voids are considered as one of the phases, i.e., as a weak material characterized by low value of Young modulus [2, 4]. The materials distribution is described by a phase field vector $\rho = \{\rho_m\}_{m=1}^s$ where the local fraction field $\rho_m = \rho_m(x) : \Omega \to R$, $m = 1, ..., s$, corresponds to the contributing phase. In order to ensure that the phase field vector describes the fractions the following pointwise bound constraints called in material science the Gibbs simplex are imposed on every ρ_m for $m = 1, 2, ..., s$,

$$\alpha_m \leq \rho_m \leq \beta_m, \quad \text{and} \quad \sum_{m=1}^{s} \rho_m = 1, \tag{1}$$

where the constants $0 \leq \alpha_m \leq \beta_m \leq 1$ are given and the summation operator is understood component wise. Moreover the total spatial amount of material fractions satisfies for $m = 1, 2, ..., s$

$$\int_\Omega \rho_m(x)dx = w_m \mid \Omega \mid, \quad 0 \leq w_m \leq 1, \quad \sum_{m=1}^{s} w_m = 1. \tag{2}$$

The parameters w_m are user defined and $\mid \Omega \mid$ denotes the volume of the domain Ω. From the equality (1) it results that $\rho_s = 1 - \sum_{m=1}^{s-1} \rho_m$ and the fraction ρ_s may be removed from the set of the design functions. Therefore from now on the unknown phase field vector ρ is redefined as $\rho = \{\rho_m\}_{m=1}^{s-1}$. Due to the simplicity and robustness the SIMP material interpolation model [4] is used. Following this model the elastic tensor $\mathcal{A}(\rho) = \{a_{ijkl}(\rho)\}_{i,j,k,l=1}^2 = \sum_{m=1}^{s} g(\rho_m)\mathcal{A}_m$ of the material body is assumed to be a function depending on the fraction function ρ:

$$\mathcal{A}(\rho) = \sum_{m=1}^{s-1} g(\rho_m)\mathcal{A}_m + g(1 - \sum_{m=1}^{s-1} \rho_m)\mathcal{A}_s, \tag{3}$$

with $g(\rho_m) = \rho_m^3$. The constant stiffness tensor $\mathcal{A}_m = \{\tilde{a}_{ijkl}^m\}_{i,j,k,l=1}^2$ characterizes the m-th elastic material of the

body. Denote by $u = (u_1, u_2)$, $u = u(x)$, $x \in \Omega$, the displacement of the body and by $\sigma(x) = \{\sigma_{ij}(u(x))\}$, $i, j = 1, 2$, the stress field in the body. Consider elastic bodies obeying Hooke's law, i.e., for $x \in \Omega$ and $i, j, k, l = 1, 2$,

$$\sigma_{ij}(u(x)) = a_{ijkl}(\rho)e_{kl}(u(x)), \tag{4}$$

where $e_{kl}(u(x)) \stackrel{def}{=} \frac{1}{2}(u_{k,l}(x)+u_{l,k}(x))$ and $u_{k,l}(x) = \frac{\partial u_k(x)}{\partial x_l}$. We use here and throughout the paper the summation convention over repeated indices [1, 2]. The stress field σ satisfies the system of equations in the domain Ω for $i, j = 1, 2$ [3]

$$-\sigma_{ij}(x)_{,j} = f_i(x) \quad \sigma_{ij}(x)_{,j} = \frac{\partial \sigma_{ij}(x)}{\partial x_j}, \tag{5}$$

The following boundary conditions are imposed: $u_i(x) = 0$ on Γ_0, $\sigma_{ij}(x)n_j = p_i$ on Γ_1, $i, j = 1, 2$ as well as nonpenetration and Tresca friction conditions on Γ_2:

$$(u_N + v) \leq 0, \quad \sigma_N \leq 0, \quad (u_N + v)\sigma_N = 0 \quad \text{on } \Gamma_2, \tag{6}$$

$$\mid \sigma_T \mid \leq 1, \quad u_T\sigma_T + \mid u_T \mid = 0 \quad \text{on } \Gamma_2. \tag{7}$$

Here $n = (n_1, n_2)$ is the unit outward versor to the boundary Γ. The normal components of the displacement and stress fields are denoted by u_N and σ_N, respectively. The tangential components of displacement u and stress σ are denoted by $(u_T)_i$ and $(\sigma_T)_i$, $i, j = 1, 2$, respectively. $\mid u_T \mid$ denotes the Euclidean norm in R^2 of the tangent vector u_T. A gap between the bodies is described by a given function v.

2.1. Phase field based topology optimization problem

The structural optimization problem for the contact problem (5)-(7) takes the form: *find* $\rho^\star \in U^\rho_{ad}$ *such that*

$$J(\rho^\star, u^\star) = \min_{\rho \in U^\rho_{ad}} J(\rho, u(\rho)), \tag{8}$$

where $u^\star = u(\rho^\star)$ denotes a solution to the state system (5)-(7) depending on $\rho^\star \in L^\infty(\Omega; R^{s-1}) \cap H^1(\Omega; R^{s-1})$. The cost functional is sum of the normal contact stress $J_\eta : H^1(\Omega) \to R$ functional and regularization functional $E(\rho) : U^\rho_{ad} \to R$:

$$J(\rho, u(\rho)) = J_\eta(u(\rho)) + E(\rho), \tag{9}$$

Functional $J_\eta(u(\rho)) = \int_{\Gamma_2} \sigma_N(u(\rho))\eta_N(x)ds$ depends on on a given auxiliary bounded function $\eta(x) \in M^{st} = \{\eta = (\eta_1, \eta_2) \in H^1(\Omega; R^2) : \eta_i \leq 0 \quad \text{on } \Omega, \ i = 1, 2, \quad \| \eta \|_{H^1(\Omega;R^2)} \leq 1\}$. Functional $E(\rho) = \sum_{m=1}^{s-1} \int_\Omega \psi(\rho_m)d\Omega$ with $\psi(\rho_m) = \frac{\gamma\epsilon}{2} \mid \nabla\rho_m \mid^2 + \frac{\gamma}{\epsilon}\psi_B(\rho_m)$ where $\epsilon, \gamma > 0$ are real constants density and function $\psi_B(\rho_m) = \rho_m^2(1 - \rho_m)^2$. Moreover $\nabla\rho_m \cdot n = 0$ on Γ for each m. The set U^ρ_{ad} of admissible fractions has the form: $U^\rho_{ad} = \{\rho \in L^\infty(\Omega; R^{s-1}) \cap H^1(\Omega; R^{s-1}) : 1 - \beta_s \leq \sum_{m=1}^{s-1} \rho_m \leq 1 - \alpha_s, \alpha_m \leq \rho_m \leq \beta_m, \ \int_\Omega \rho_m dx = w_m \mid \Omega \mid \quad \text{for } m = 1, ..., s-1\} \neq \emptyset$.

2.2. Necessary optimality condition

The Lagrange multipliers method has been used to calculate the directional derivative of the cost functional (9) with respect to the function ρ and to formulate necessary optimality condition to problem(8). Taking into account that the structural optimization problem (8) can be considered as a phase transition setting problem the constrained gradient flow equation of Allen-Cahn for the cost functional (9) type is formulated and numerically solved.

3. Numerical method

Operator splitting approach [4] has been used to solve Allen-Cahn equation numerically. First the trial value of the optimal solution is calculated for the unconstrained gradient flow equation for these parts of the cost functional which are defined either in domain Ω or on boundary Γ_2. Next this solution is updated by solving the gradient flow equation for curvature term of cost functional (9) only and enforcing the constraints by the projection operation.

4. Numerical results

The topology optimization problem (8) has been discretized and solved numerically. Time derivatives are approximated by the forward finite difference. Piecewise constant and piecewise linear finite element method is used as discretization method in space variables. The derivative of the double well potential is linearized with respect to ρ_m. Primal-dual active set method has been used to solve the state system. Fig. 1 presents the optimal topology domain obtained by solving structural optimization problem (8) using the formulated gradient flow Allen-Cahn equation. The areas with the weak phases appear in the central part of the body and near the fixed edges. The areas with the strong phases appear close to the contact zone and along the edges. The rest of the domain is covered with the intermediate phase. The obtained normal contact stress for the optimal topology is almost constant along the contact boundary and has been significantly reduced comparing to the initial one.

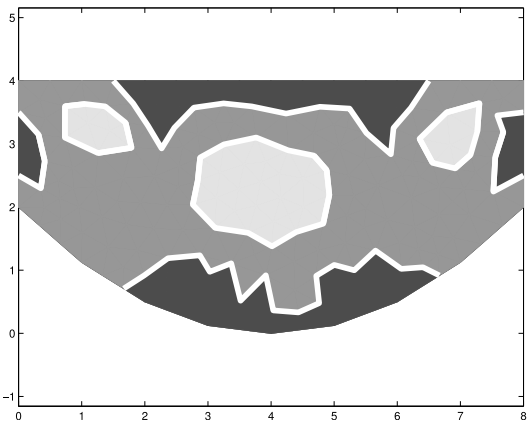

Figure 1: Optimal material distribution in domain $\Omega^\star$.

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