A comparison of free vibration frequencies for periodic Rayleigh and Timoshenko beams. Tolerance modelling

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Abstract

The paper deals with linear-elastic Rayleigh and Timoshenko beams with periodic structure. The governing equations with highly oscillating coefficients are converted into systems of differential equations with constant coefficients. The proposed tolerance models are obtained in the framework of the tolerance modelling and studied numerically using Galerkin's method. An analysis of a simply supported periodic beam is performed. Natural linear frequencies are determined for both the kinds of beams - Rayleigh and Timoshenko and analysed within the obtained models.

Keywords: periodic beams, Rayleigh and Timoshenko beams, the effect of the mesostructure size, tolerance modelling

1. Introduction

Vibrations of beams with periodically varying geometric and material properties along the beam axis are considered here, cf. Fig. 1. These beams are consisted of many small identical elements, called *periodicity cells*. In this note, two models of such beams are considered: 1) for slender Rayleigh-type beams, where the length of the cell l is assumed to satisfy conditions $d<<l<<L$ (d is the height of the beam, L is the span of the beam), 2) for shear-deformable Timoshenko-type beams, where the length of the cell l satisfies condition $l<<L$ and height of the beam d satisfies condition $d/l<<1/4$.

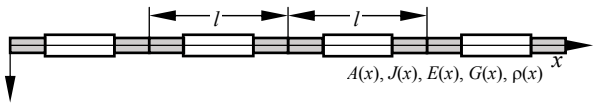

Figure 1: A fragment of periodic beam

- *The slender Rayleigh-type beam.*

The first model is based on the assumptions of Rayleigh-type beams. An analysis is restricted to bending vibrations. Denoting by w the deflection, and by ρ, A, J, E - the mass density, the cross-section area of the beam, the area moment of the inertia, the Young's modulus, the governing equation for free vibrations of these beams can be written in the form:

$$\partial\partial(EJ\partial\partial w)+\rho A\ddot{w}-\partial(\rho J\partial\ddot{w})=0. \tag{1}$$

Equation (1) has highly oscillating, non-continuous, periodic coefficients in x.

- *The shear-deformable Timoshenko-type beam.*

The second model is based on the assumptions of the Timoshenko-type beams. Denote the deflection of the beam by w, the cross-section rotation by θ, and introduce G as the Kirchhoff's modulus and k as the shear modification coefficient. The governing equations for free vibrations of such beams can be written in the form:

$$\begin{aligned} &kGA(\partial\partial w-\partial\theta)-\rho A\ddot{w}=0,\\ &EJ\partial\partial\theta+kGA(\partial w-\theta)-\rho J\ddot{\theta}=0, \end{aligned} \tag{2}$$

where all coefficients can be highly oscillating, non-continuous, periodic functions of x.

Because Equations (1) and (2) have periodic, functional coefficients, they are not good tools to solve some special problems of these beams. Hence, various averaging methods are proposed to obtain equations with constant coefficients. Most of them is based on the asymptotic homogenization, Ref. [1].

Dynamic problems of periodic beams are analysed in many papers, e.g.: frequency band gaps using the differential quadrature method in Ref. [14]; a flexural wave propagation in the beam on elastic foundation in Ref. [15]; a periodic Euler-Bernoulli beam in Ref. [2]; the two-scale asymptotic expansion method is applied in Ref. [6]. However, governing equations of these models neglect often the effect of the mesostructure size. In order to take into account this effect *the tolerance modelling* is applied, proposed and developed for periodic and non-periodic media with a microstructure, Ref. [13].

Using this method governing equations with highly oscillating, non-continuous, periodic or non-periodic coefficients can be replaced by equations with constant or slowly-varying coefficients. Applications of this approach to

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some dynamic problems of various periodic structures are shown in a series of papers, e.g.: for slender beams Ref. [10], Ref. [11]; for nonlinear vibrations of slender beams [3, 4, 5]; for sandwich plates Ref. [9]; for Timoshenko beams Ref. [12]; for nonlinear vibrations of thin viscoelastic plates Ref. [7]. The tolerance modelling is also used to analyse some problems of non-periodic structures, e.g.: microstructured plates Ref. [8].

The main aims of this contribution is to show and apply *the tolerance model equations for slender Rayleigh-type beams* and *for Timoshenko-type beams*, describing the effect of the mesostructure size, and compare obtained results.

2. Modelling assumptions

In the tolerance modelling some introductory concepts are used, e.g.: the slowly-varying function, the fluctuation shape function, the averaging operator, Ref. [13]. Moreover, this method is based on two fundamental assumptions.

The first is the assumption of *micro-macro decompositions* for basic unknowns, i.e.:

- for the slender Rayleigh-type beams the unknown deflection w can be decomposed as: $w(x,t)=W(x,t)+h^A(x)V^A(x,t)$, $A=1,..,N$, where basic unknowns - macrodeflection W and amplitudes of fluctuation variables V^A are slowly-varying functions of x;
- for the shear-deformable Timoshenko-type beams the deflection w, the cross-section rotation θ, can be decomposed as: $w(x,t)=W(x,t)+h^A(x)V^A(x,t)$, $\theta(x,t)=\Theta(x,t)+p^R(x)Z^R(x,t)$, $A=1,..,N$, $R=1,\ldots,M$, where basic unknowns - W, Θ (macrodeflection, macrorotation) and V^A, Z^R (amplitudes of fluctuation variables) are slowly-varying functions of x.

The second assumption is *the tolerance averaging approximation*, i.e. some terms are assumed to be negligible small.

3. Model equations

Using the introductory concepts, the fundamental assumptions and the tolerance modelling procedure, cf. Ref. [13], after some manipulations, Eqn (1) and Eqns (2) can be replaced by equations with constant coefficients:

- *the tolerance model equations for the Rayleigh-type beams* take the from:

$$
\begin{aligned}
&< EJ > \partial\partial\partial\partial W + < EJ\partial\partial h^A > \partial\partial V^A + < \rho A > \ddot{W} - \\
&\quad - < \rho J > \partial\partial\ddot{W} - \underline{< \rho J\partial h^A >}\partial\ddot{V}^A = 0, \\
&< EJ\partial\partial h^A > \partial\partial W + \underline{< \rho J\partial h^A >}\partial\ddot{W} + \\
&\quad + < EJ\partial\partial h^A\partial\partial h^B > V^B + \\
&\quad + \underline{< \rho A h^A h^B >}\ddot{V}^B + \underline{< \rho J\partial h^A\partial h^B >}\ddot{V}^B = 0;
\end{aligned}
\tag{3}
$$

- *the tolerance model equations for the Timoshenko-type beams* have the from:

$$
\begin{aligned}
&< kGA > (\partial\partial W - \partial\Theta) + < kGA\partial h^A > \partial V^A - \\
&\quad - \underline{< kGA\partial h^A p^R >}\partial Z^R - < \rho A > \ddot{W} - \underline{< \rho A h^A >}\ddot{V}^A = 0, \\
&< kGA\partial h^A > (\partial W - \Theta) + < kGA\partial h^A\partial h^B > V^B - \\
&\quad - \underline{< kGA\partial h^A p^R >}Z^R + \\
&\quad + < \rho A h^A > \ddot{W} + \underline{< \rho A h^A h^B >}\ddot{V}^B = 0, \\
&< EJ > \partial\partial\Theta + \underline{< EJ\partial p^R >}\partial Z^R + < kGA > (\partial W - \Theta) + \\
&\quad + < kGA\partial h^A > V^A - \underline{< kGAp^R >}Z^R - \\
&\quad - < \rho J > \ddot{\Theta} - \underline{< \rho J p^S >}\ddot{Z}^R = 0, \\
&< EJ\partial p^R > \partial\Theta + < EJ\partial p^R\partial p^S > Z^S - \\
&\quad - \underline{< kGAp^R >}(\partial W - \Theta) - \underline{< kGA\partial h^A p^R >}V^A + \\
&\quad + \underline{< kGAp^R p^S >}Z^S + \underline{< \rho J p^R >}\ddot{\Theta} + \underline{< \rho J p^R p^S >}\ddot{Z}^S = 0.
\end{aligned}
\tag{4}
$$

In the above Eqns (3)-(4) underlined terms depend on the mesostructure parameter l.

4. Remarks

The tolerance model equations for various assumptions of periodic beams can be applied to analyse and compare the effect of the mesostructure size on free vibrations of these beams.

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