Velocity-based monolithic approach to fluid-structure interaction problem: formulation, stabilization and preconditioning strategy*

Michał Wichrowski[1], Piotr Krzyżanowski[2], Stanisław Stupkiewicz[1]

[1] *Insitute of Fundamental Technological Reaserch, Polish Academy of Sciences, Pawinskiego 5B, 02-106 Warsaw, Poland*
[2] *University of Warsaw, Faculty of Mathematics, Informatics and Mechanics, Banacha 2, 02-097 Warsaw, Poland*
e-mail: mwichro@ippt.pan.pl

Abstract

Solution methods for fluid-structure interaction (FSI) problems are considered. For time integration, a semi-implicit scheme is used allowing to decouple momentum and geometry. Momentum balance, formulated as a velocity-based saddle-point problem, is solved using a monolithic approach. Finite-element discretization of the system is applied, and a new nonuniform stabilization is proposed. Robust block preconditioners based on uniform well-posedness are discussed and verified through numerical experiments. Convergence of the presented method is verified numerically on Turek's benchmark.

Keywords: fluid-structure interaction, preconditioning, monolithic scheme

1. Introduction

When an elastic body immersed in a fluid is flexible enough, interaction between the flow and the structure has a crucial influence on the physics of the phenomenon. It is then not possible to neglect the influence of (not known *a'priori*) variable domain geometry or to solve decoupled equations describing the flow and the deformation of the structure.

Two general approaches are used for simulating FSI problems: partitioned schemes, where the solid and fluid equations are solved separately, and monolithic schemes, where the complete set of coupled equations is solved simultaneously. A partitioned approach requires a coupling algorithm to enforce coupling conditions but allows reusing existing codes. A monolithic approach needs a specialized code, but it is more favorable for its stability.

In this work, the monolithic approach proposed in [1] is further developed. A semi-implicit algorithm is used for time integration: momentum equation is solved implicitly, while geometry changes are treated explicitly (geometry-convective explicit scheme). The resulting scheme exhibits good temporal stability, allowing time steps of the order of 0.01 s at $\mathrm{Re} = 200$ for Turek's benchmark [2]. Moreover, the problem solved at each time step is linear. We propose optimal block preconditioners [3, 4] for the related iterative solver.

2. Derivation of the fluid-structure interaction model

The FSI equation is reformulated to a Stokes-like problem with variable viscosity. In [1], interaction of the fluid with a linear elastic solid have been analysed. This approach is here extended for an incompressible hyperelastic solid.

The FSI model is derived starting from Cauchy momentum balance for the fluid (subscript f) and the solid (subscript s) within the domain $\Omega = \Omega_f \cup \Omega_s$:

$$\begin{aligned} \rho_f \frac{\mathrm{D}v_f}{\mathrm{D}t} - \nabla \cdot \sigma_f = g_f \qquad & \text{in } \Omega_f, \\ \rho_s \frac{\mathrm{D}v_s}{\mathrm{D}t} - \nabla \cdot \sigma_s = g_s \qquad & \text{in } \Omega_s, \end{aligned} \tag{1}$$

where $\frac{\mathrm{D}}{\mathrm{D}t}$ denotes the material derivative. Coupling between the solid and the fluid is expressed through interface conditions:

$$v_f = v_s, \qquad \sigma_f \cdot n_f = \sigma_s \cdot n_s. \tag{2}$$

One can observe that the no-slip condition $(2)_1$ is equivalent to continuity of velocity. This can be used to define a unified velocity field,

$$v(x) = \begin{cases} v_s(x) & x \in \Omega_s, \\ v_f(x) & x \in \Omega_f. \end{cases} \tag{3}$$

By rearranging the local momentum balance (1) in a weak form, the second interface condition $(2)_2$ can be eliminated, and the momentum balance is then reformulated as follows:

$$\int_\Omega \rho \frac{\mathrm{D}v}{\mathrm{D}t} \phi \mathrm{d}x + \int_\Omega \sigma : \nabla \phi \mathrm{d}x = \int_\Omega g \phi \mathrm{d}x + \int_{\Gamma_N} (n \cdot \sigma) \phi \mathrm{d}s. \tag{4}$$

By introducing the arbitrary Eulerian-Lagrangian (ALE) description, the material derivative is expanded as:

$$\frac{\mathrm{D}v}{\mathrm{D}t} = \frac{\mathrm{d}v}{\mathrm{d}t} + (v - v_A)\nabla v, \tag{5}$$

where v_A is the mesh velocity, and the derivative $\frac{\mathrm{d}v}{\mathrm{d}t}$ is evaluated holding the mesh point fixed.

For completeness, fluid and solid models are needed. Here, we consider an incompressible Newtonian fluid,

$$\sigma_f = 2\mu_f \epsilon(v_f) - p_f I, \qquad \nabla \cdot v_f = 0, \tag{6}$$

where p is the pressure, and μ_f is the viscosity. The solid is modelled as an incompressible neo-Hookean material,

$$\hat{\sigma}_s = \mu_s \hat{F}\hat{F}^T - \hat{p}_s I, \qquad \nabla \cdot v_s = 0, \tag{7}$$

where $\hat{F} := \hat{\nabla}\Phi_s = I + \hat{\nabla}\hat{u}_s$ is the deformation gradient, $\hat{p}$ is the pressure, and μ_s is the shear modulus. Quantities with a superimposed hat refer to fields defined in the reference configuration. This results in the following weak form:

$$\begin{cases} a(v,\phi) + b(\phi,p) = g(\phi) & \forall \phi \in H^1_{D0}(\Omega), \\ b(v,q) = 0 & \forall q \in L^2(\Omega), \\ \partial_t \hat{u}_s = \hat{v}, \end{cases} \tag{8}$$

* This work was partially supported by the National Science Centre (NCN) in Poland through grant no. 2015/19/N/ST8/03924.

where

$$a(v,\phi) = (\rho\frac{\mathrm{D}v}{\mathrm{D}t},\phi)_\Omega + (2\mu_f\epsilon(v_f) : \epsilon(\phi_f))_{\Omega_f} + (\mu_s\hat{F},\nabla\hat{\phi}_s)_{\Omega_s}, \quad (9)$$

$$b(v,q) = (\nabla\cdot v, q)_\Omega.$$

A semi-implicit time discretization is next introduced. The geometry is computed explicitly while the momentum balance is solved implicitly. The material derivative $\frac{\mathrm{D}v}{\mathrm{D}t}$ is discretized using a semi-implicit formula for obtaining a linear equation to be solved at each time step.

Well-posedness of the problem solved at an individual time step is analysed for specially chosen norms by verification of the inf–sup conditions. Details are omitted here.

2.1. *Time integration algorithm*

The solution procedure proceeds incrementally, and at each time step the following three steps are preformed:

- **Explicit part:** New geometry is computed explicitly using the data from the previous time step.
- **Implicit part:** Using the new geometry, the momentum balance, $(8)_1$–$(8)_2$ is solved implicitly, and new values of velocity and pressure are thus obtained.
- **Postprocess the solution:** The new displacement is computed using the displacement from the previous time step and the velocity obtained in the current time step.

2.2. *Finite-element model*

Finite-element discretization yields a fully-discrete problem. However, the discretized equations require stabilization to remain well-posed independently of problem parameters. The *grad-div* stabilization is introduced as follows:

$$\begin{cases} \tilde{a}(v,\phi) + b(\phi,p) = f(\phi), \\ b(v,q) = 0, \end{cases} \quad (10)$$

where

$$\tilde{a}(v,\phi) = a(v,\phi) + (r\nabla\cdot v, \nabla\cdot\phi)_\Omega. \quad (11)$$

Two versions of stabilization are considered: constant r, as used in [1], and variable r, as proposed in this work. Numerical experiments shows that variable stabilization tailored to local material parameters improves the quality of the solution.

3. Results

The tests are performed on Turek's benchmark [2]. The geometrical setting with a coarse mesh is shown in Fig. 1.

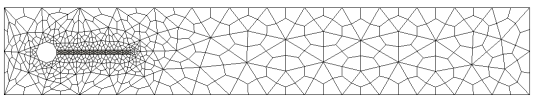

Figure 1: Coarse mesh for Turek's benchmark problem

The results obtained using different stabilization methods are shown in Fig. 2.

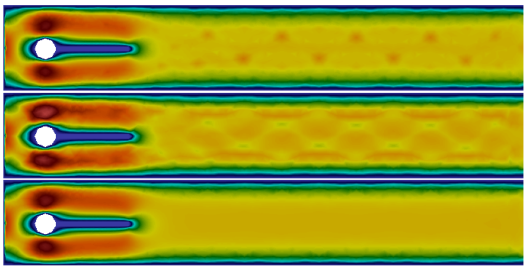

Figure 2: Comparison of stabilization methods for Turek's benchmark problem: no stabilization (top), constant stabilization proposed by Xu[1] (middle) and variable stabilization (bottom)

For solving the system of linear equations, the preconditioned Krylov method has been used. Diagonal and upper-triangular preconditioners have been tested for both uniformly and nonuniformly stabilized formulation.

4. Conclusion

The proposed time integration algorithm consists of a few simple steps that are easy to implement. Stabilization and solution methods for the implicit part of the algorithm have been developed and implemented. Nonuniform stabilization results in smoother results and allows to avoid numerical artifacts appearing at mesh irregularities. Numerical experiments illustrate the robustness of the applied solution methods. The number of Krylov iterations required for both stabilization methods are comparable.

References

[1] Xu, J. and Yang, K., Well-posedness and robust preconditioners for discretized fluid–structure interaction systems *Comput. Method. Appl. M.* 292, pp. 69-91, 2015.

[2] Turek S. and Hron J., Proposal for numerical benchmarking of fluid-structure interaction between an elastic object and laminar incompressible flow. In: *Bungartz HJ., Schafer M. (eds) Fluid-Structure Interaction. Lecture Notes in Computational Science and Engineering*, vol 53, pp. 371-385, Springer, Berlin, Heidelberg, 2006.

[3] Krzyżanowski, P., On block preconditioners for saddle point problems with singular or indefinite (1, 1) block. *Numer. Linear Algebra Appl.* 18 pp. 123-140, 2011.

[4] Mardal, K. A., and Winther, R. , Preconditioning discretizations of systems of partial differential equations. *Numer. Linear Algebra Appl.* 18 pp. 1-40, 2011.