Numerical analysis of ellipticity condition for large strain plasticity

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Abstract

This paper deals with the numerical investigation of ellipticity of the boundary value problem for finite strain elasto-plasticity which can be lost when softening occurs. A discontinuity surface appears then in the material and, from the theoretical point of view, this leads to ill-posedness of the boundary value problem. In the paper the analysis for the isothermal conditions is first carried out. The ellipticity condition is tested using the acoustic tensor which can be computed in several ways depending on the stress and strain measures considered. Next, the thermomechanical coupling is approached which requires the formulation of additional conditions related to the thermal field. The verification of ellipticity is performed numerically during Finite Element Method computations.

Keywords: ellipticity, acoustic tensor, large strains, plasticity, thermo-plasticity

1. Introduction

When a material experiences a load which grows to extreme values it initially deforms uniformly and from some point of the process strains increase only in a certain zone (band) while the rest of the sample experiences unloading. This phenomenon called strain localization is related to the occurrence of a discontinuous bifurcation. It means that in the material a jump of the velocity gradient across a certain surface can appear. The classical local continuum theory fails to describe the problem correctly, since in this situation the boundary value problem is ill-posed and the elliptic equations become hyperbolic [1].

The aim of this paper is to investigate the ellipticity of the governing equations for an elasto-plastic material in isothermal and non-isothermal conditions, undergoing strain localization caused by different types of softening: material, thermal and geometrical. The analysis is performed at the chosen Gauss points after convergent load steps during Finite Element Method (FEM) calculation.

2. Ellipticity for large strain isothermal models

In the reference configuration the traction equilibrium for the stationary discontinuity surface Σ is defined using the rate of the first Piola-Kirchhoff stress tensor $\mathbf{P}$ [2]

$$[\![\dot{\mathbf{P}}]\!]\mathbf{N} = \mathbf{0} \tag{1}$$

where $[\![\cdot]\!]$ denotes the jump of a quantity. Vector $\mathbf{N}$ is normal to the discontinuity surface in the reference configuration, see Figure 1.

The following non-zero jump of the velocity gradient across the discontinuity surface is admitted

$$[\![\mathbf{l}]\!] = \mathbf{g} \otimes \mathbf{n} \neq \mathbf{0} \tag{2}$$

where $\mathbf{n}$ is a vector normal to the discontinuity surface in the current configuration, see Figure 1, related to $\mathbf{N}$ through the formula $\mathbf{n} = [\mathbf{F}^{-\mathrm{T}}\mathbf{N}]/|\mathbf{F}^{-\mathrm{T}}\mathbf{N}|$ and $\mathbf{g}$ is a vector describing the jump.

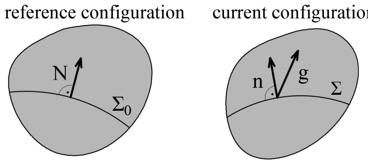

Figure 1: Discontinuity surface in reference and current configuration

Using incrementally linear constitutive relation between the rates of first Piola-Kirchhoff stress tensor and the deformation gradient $\dot{\mathbf{P}} = \mathbf{D}^{PF} : \dot{\mathbf{F}}$, and applying the relation $\dot{\mathbf{F}} = \mathbf{l}\mathbf{F}$, the following condition is obtained

$$\left[\mathbf{D}^{PF} : [\mathbf{g} \otimes \mathbf{N}]\right]\mathbf{N} = \mathbf{0} \tag{3}$$

The above equation has a non-trivial solution if the determinant of the acoustic tensor defined in the following equation is zero

$$Q_{ij} = D^{PF}_{iKjL} N_K N_L \tag{4}$$

The acoustic tensor formulated for $\mathbf{P}$ and $\mathbf{F}$ has an analogous form as for the small strain analysis, see e.g. [1], however, it can be noticed that $\mathbf{P}$ and $\mathbf{F}$ are nonsymmetric. The acoustic tensor can be also formulated for symmetric stress and strain tensors: in reference configuration the second Piola-Kirchhoff stress and the right Cauchy-Green deformation tensors can be used with the relation between them $\dot{\mathbf{S}} = \mathbf{D}^{SC} : \dot{\mathbf{C}}$ and then

$$Q_{ij} = \frac{1}{2}\left[N_K S_{KL} N_L\right]\delta_{ij} + N_L F_{iK} D^{SC}_{KLMN} F_{jM} N_N \tag{5}$$

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Alternatively, in the current configuration the acoustic tensor has the following form

$$Q_{ij} = \frac{1}{2}\left[n_k \sigma_{kl} n_l\right]\delta_{ij} + n_k D^{\sigma d}_{ikjl} n_l \tag{6}$$

where σ_{kl} are the components of the Cauchy stress tensor and tangent $\mathbf{D}^{\sigma d}$ is a spatial counterpart of tangent $\mathbf{D}^{SC}$.

It is worth mentioning that, alternatively to the jump conditions, a stability analysis can be performed using wave propagation approach, for the detailed explanation see e.g. [1].

3. Analysis for thermomechanical models

For thermomechanical coupling additional requirements for jumps of the heat flux (rate) to be zero across the discontinuity surface in the current configuration are formulated

$$[\![\mathbf{q}]\!]\cdot\mathbf{n} = 0, \qquad [\![\dot{\mathbf{q}}]\!]\cdot\mathbf{n} = 0 \tag{7}$$

with spatial continuity of the temperature field $[\![T]\!] = 0$. Assuming a stationary discontinuity surface, the jump of temperature rate can be non-zero, see [3]. Further, the thermomechanical analysis of instabilities leads to two equations resulting from Equation (1) and the balance of energy which involves the jump of rate of temperature and the jump of divergence of the heat flux. A detailed discussion of stability conditions for thermo-inelastic materials is included in [3].

4. Elasto-plasticity with thermomechanical coupling

The verification of ellipticity condition is performed for the isotropic elasto-plastic material in isothermal and nonisothermal conditions. Here, the thermomechanical model is succinctly presented.

The description is based on the multiplicative decomposition of the deformation gradient into thermal, elastic and plastic parts and on a suitable free energy potential, c.f. [4]. Associative rate-independent plasticity with Huber-Mises-Hencky yield criterion is considered.

Thermomechanical coupling involves thermal expansion, heat generation during plastic deformation and dependence of the yield strength on temperature change (thermal softening). The Fourier heat equation for the current configuration is applied for the transient process. A detailed description of the analysed model and its implementation within the Finite Element Method are included in [5].

5. Numerical results

The ellipticity condition is numerically verified for two- and three-dimensional cases. For the former one, the plane strain conditions are assumed. The material tangent is computed at the level of a Gauss point, using automatic differentiation available in *AceGen* package [6], and saved in a data base for further analysis in *AceFEM* environment. For plane strain analysis the acoustic tensor is computed for vectors $\mathbf{N} = [\cos\alpha, \sin\alpha, 0]$ where angle $\alpha \in [0, \pi]$ is discretized using a specified increment. In the three-dimensional case the analysed vectors depend, in turn, on two discretized angles: $\mathbf{N} = [\cos\alpha\cos\beta, \sin\alpha\cos\beta, \sin\beta]$. Alternatively, the minimization of $\det(\mathbf{Q})$ with respect to angles α and β, presented in [7], can be used.

In Figure 2 a response of an elongated rectangular plate with imperfection in plane strain conditions is presented. The applied material model is ideal elasto-plasticity. The decreasing load-displacement diagram and necking observed in the deformed mesh result from geometrical softening.

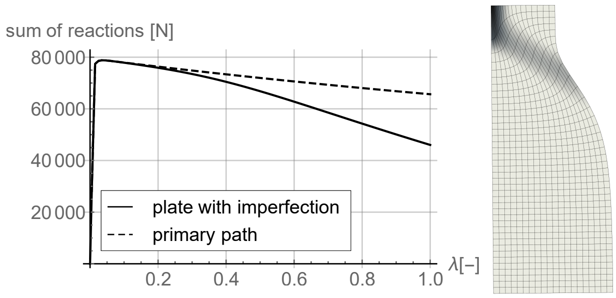

Figure 2: Sum of reactions vs enforced displacement multiplier and deformed mesh for elongated plate in plane strain

The analysis of the ellipticity condition is performed for the Gauss point in the imperfect element. The minimum value of the determinant of the acoustic tensor versus enforced displacement multiplier is presented in Figure 3. It can be observed that in the initial phase of the plastic response the determinant of $\mathbf{Q}$ reaches negative values providing a confirmation of the loss of ellipticity. The analysis of the critical directions in the reference and current configurations reveals that the normal vector $\mathbf{n}$ in the deformed specimen is inclined about 45 deg during the whole deformation process.

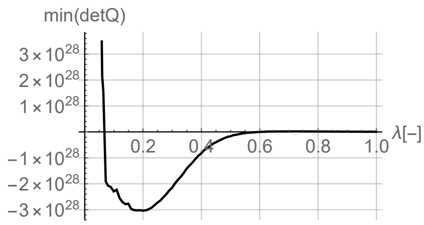

Figure 3: Minimum value of the determinant of acoustic tensor vs enforced displacement multiplier

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