Structural optimization with shapes parameterization by the assumed energy density on the structural surface

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Abstract

The trabecular bone adapts its form to mechanical loads and is able to form lightweight but very stiff structures. In this sense, it is a problem (for the Nature) similar to the structural optimization, especially the topology optimization. The structural optimization system based on shape modification using shape derivative will be presented. In general the problem of stiffest design (compliance minimization) has no solution. If the volume of the object is increasing, the compliance is decreasing. Thus, in the standard approach to the energy based topology optimization the additional constraint has to be added. Usually the volume of the material is limited. But with such an assumption the optimization procedure does not include any criteria for stress. In case of the biomimetic approach presented here, the role of the additional constraint plays the strain energy density on the structural surface (which is between two assumed levels forming the insensitivity zone) and the volume or structural mass results from the optimization procedure.

Keywords: shape optimization, topology optimization, trabecular bone remodelling

1. The trabecular bone remodelling phenomenon as a biological pattern for optimization

While examining biological structures, we often realize that they must be optimal from both mechanical and mathematical optimality perspectives. The trabecular bone is here an excellent example. Wolff's law [1], formulated in 19th century assumes that bone is capable of adapting to mechanical stimulation and optimizing energy expenditure to keep tissue in good condition. This aspect could be useful when issues of structural optimization are discussed. Healthy tissue of trabecular bone has very sophisticated shape. The tissue forms a network of beams called trabeculae. This structure is able to handle a wide range of loads being continually rebuilt. In many numerical models of trabecular bone remodeling strain energy density (SED) is used to measure the level of mechanical stimulation. According to biomechanical models [2, 3, 4], the phenomenon of trabecular bone adaptation, called remodelling, has two important attributes. First, mechanical stimulation is necessary to conserve rebuilding balance. Second, the process of resorption and formation occurs only on the bone surface. In this way the bone reacts to external forces and the process of remodeling leads to mechanical adaptation. The model of the bone remodelling process discussed in the paper is based on the regulatory model of Huiskes. The main idea of the model consists of a regulatory mechanism (on the bone surface only) between bone resorption and formation. Bone mass increases above a certain level of mechanical stimulation (measured by the strain energy density on the tissue surface) and decreases below the other certain energy level. When the strain energy density on the structural surface is between these two levels the bone mass is maintained and this rate of bone mechanical stimulation is called adaptation or lazy zone (insensitivity zone), because then the bone does not react to the changes in mechanical stimulation level. It is interesting, that SED, as energy measure, is also emphasized in optimization research, distant from biomechanical studies [5, 6, 7], where one can find the theorem, that for the stiffest design the energy density along the shape to be designed must be constant.

2. The structural stiffness maximization

The goal is to maximize the stiffness of a structure, that is minimization of the functional

$$J(\Omega) = \int_{\Gamma_1} \mathrm{t.u}\, ds \tag{1}$$

under constraints

$$\int_{\Omega} dx - V_0 = 0 \tag{2}$$

and state equations

$$\mathrm{div}\sigma(\mathrm{u}) = 0 \quad \text{in} \quad \Omega \tag{3}$$

$$\sigma(\mathrm{u}).\mathrm{n} = \mathrm{t} \quad \text{on} \quad \Gamma_1 \tag{4}$$

$$\sigma(\mathrm{u}).\mathrm{n} = 0 \quad \text{on} \quad \Gamma_v \tag{5}$$

$$\mathrm{u} = 0 \quad \text{on} \quad \Gamma_0 \tag{6}$$

Here, Ω represents the domain of the elasticity system, u the displacement, V_0 a given volume, Γ_0 part of the boundary with

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Dirichlet condition, Γ_1 part of the boundary loaded by traction forces, Γ_v part of the boundary subject to modification.

To justify an assumption about constant energy density on the structural surface the Lagrangian for the problem under considerations is defined in the form

$$L(\Omega_t, \lambda) = \int_{\Gamma_1} t.u \, ds + \lambda \left[\int_{\Omega_t} dx - V_0 \right] \tag{7}$$

The Lagrange function depends only on two scalar variables (t, λ). Then the shape differentiation of both Lagrange function and weak state equation using speed method is performed [8, 9, 10]. At the local minimum this first derivative should vanish. For the stationary point for the part of the boundary subject to modification

$$\sigma(u) : \varepsilon(u) = \lambda = \text{const.} \tag{8}$$

Of course the value of λ is not known. The assumed value of the strain energy density on the part of the boundary subject to modification could be related to the material properties. Change in the assumed value of the strain energy density results in change of the structural form - topology and volume. In this way, the final structural volume results from the optimization procedure. Instead of imposing volume constraint it is possible to parameterize shapes by the assumed energy density on the structural surface, which may be quite accurately predicted from the yield criteria.

The biomimetic, heuristic algorithm for the structural optimization with shapes parameterization by the assumed energy density on the structural surface can be described as follows [11, 12]:

– it is assumed that the energy density $\sigma(u) : \varepsilon(u)$ has a constant λ value on Γ_v ,

– if at a given point on Γ_v this density is bigger than λ + s then the boundary in moved outside,

– if at a given point on Γ_v this density is smaller than λ − s then the boundary in moved inside,

– these steps are repeated until equilibrium is achieved,

– the value of λ is modified if the final design is unsatisfactory.

3. Conclusions

In the paper the structural optimization with shapes parameterization by the assumed energy density on the structural surface was presented. The stiffest design is obtained by adding or removal material on the structural surface in the virtual space. The assumed value of the strain energy density on the part of the boundary subjected to modification could be related to the material properties. Change in the assumed value of the strain energy density results in change of the structural form - topology and volume. In this way, the final structural volume results from the optimization procedure. Instead of imposing volume constraint the shapes are parameterized by the assumed energy density, which may be quite accurately predicted from the yield criteria. The remodelling phenomenon with the lazy zone and strain energy density equalization on the trabecular bone surface assumptions could be described from the stiffness point of view. The functional configurations during the process of optimization allow including size, shape and topology optimization in one numerical procedure. Instead of the assumption of a constant volume, the assumption of a constant strain energy density on the surface of the structure is used and the volume or structural mass results from the optimization procedure. Therefore, presented in the paper approach can be used as a method of structural optimization unrelated already to trabecular bone remodelling phenomenon.

References

[1] Wolff J., Das Gesetz der Transformation der Knochen, Hirschwald, 1892.

[2] Huiskes R., Ruimerman R., Van Lenthe G.H. and Janssen J.D., Effects of mechanical forces on maintenance and adaptation of form in trabecular bone, *Nature* 405, pp. 704-706, 2000.

[3] Huiskes R., Weinans H., Grootenboer J., Dalstra M., Fudala M., Slooff T.J., Adaptive bone remodelling theory applied to prosthetic-design analysis, *J. Biomech*. 20, pp. 1135-1150, 1987.

[4] Mullender M.G., Huiskes R., Proposal for the regulatory mechanism of Wolff's law, *J. Orthop.* Res. 13(4), pp. 503-512, 1995.

[5] Mroz Z., On a problem of minimum weight design", *Quast. Appl. Math.* 19, pp. 127-135, 1961.

[6] Pedersen P., Optimal Designs - Structures and Materials - Problems and Tools, Department of Mechanical Engineering, Solid Mechanics, Technical University Denmark, 2003.

[7] Wasiutynski Z., On the congruency of the forming according to the minimum potential energy with that according to equal strength", *Bull. de l'Academie Polonaise des Sciences, Serie des Sciences Techniques* 8(6), pp. 259-268, 1960.

[8] Plotnikov, P., Sokołowski, J., Compressible Navier-Stokes equations. Theory and Shape Optimization, Birkhauser/Springer Basel AG, Basel, 2012.

[9] Sokołowski J., Zolesio J., Introduction to Shape Optimization. Shape Sensitivity Analysis, Springer-Verlag, 1992.

[10] Sokołowski J., Żochowski A., On the Topological Derivative in Shape Optimization *SIAM J. Control Optim*., 37(4), pp. 1251–1272, 1999.

[11] Nowak M., Structural optimization system based on trabecular bone surface adaptation", *Struct Multidisc Optim* 32, pp. 241-251, 2006.

[12] Nowak M., On Some Properties of Bone Functional Adaptation Phenomenon Useful in Mechanical Design, *Acta of Bioengineering and Biomechanics*, Vol. 12, No.2, pp. 49-54, 2010.