A mixed finite element formulation for finite elasticity of solids with two-fibre family reinforcement

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Abstract

We propose a mixed finite element formulation to solve the boundary-value problem of a finite elastic body reinforced by two equal distinct, possibly stiff fibre families. The formulation is based on splitting both the deformation and the stress field into the uniaxial stretches and tensions along the preferred directions and the remaining deformation and stress, respectively. The stretches and tensions are the auxiliary 4 fields to be approximated besides the displacements. The resulting 5-field nonlinear mixed formulation is solved with the Newton-Raphson procedure. We demonstrate application of error estimation and adaptivity.

Keywords: anisotropic, two fibres, adaptivity, inextensible, finite elements

1. Introduction

Strongly anisotropic elastic materials play an essential role in many fields including mechanics of composits, mechanics of soft biological tissues and many others. A frequent source of anisotropy is the presence of reinforcing fibres which display strong stiffening properties accompanying their stretching. In the case of soft tissues this role is played by the collagen fibres which show strong, even exponential stiffening under moderate stretch. This phenomenon appears as near inextensibility and it is in some way similar to near incompressibility which is observed for the ruber-like materials. Numerical modeling of a nearly inextensible material by the Finite Element Method (FEM) may cause similar difficulties as approximation of a nearly incompressible solid body: unstable or oscilatory solutions. The remedy for incompressible mechanics is the well known splitting of the description of kinematics into the volumetric part (dilatation) and the unimodular deformation gradient. It is accompanied by the corresponding split of the stress into the energy conjugate parts: the spherical contribution of the pressure and the deviatoric stress. In addition we use a separate approximation for the displacements $\boldsymbol{u}$, the dilatation θ and the pressure p. The mixed formulation involving these 3 fields with adequate approximation in the Q^p and P^{p-1} finite element spaces for the displacements $\boldsymbol{u}$ and the auxiliary variables θ and p, respectively, allows one for effective modeling of the nearly incompressible solid [3].

In our previous works we developed a sort of adaptation of this approach for the materials with near inextensibility in one preferred direction [2], possibly accompanied with incompressibility [1]. The present work is devoted to anisotropic materials reinforced with two families of fibres, that is with two preferred directions whose stretches cause rapid stiffening in these directions. Materials of this kind constitute, for instance, soft tissues of arteries in which case the two directions form two spiral lines around the artery.

2. Description of kinematics and stresses

In this section we briefly present the main principles of constructing the mixed formulation for the materials reinforced with two families of fibres. We assume that the two preferred directions of reinforcement are given by two distinct fields of unit vectors $\boldsymbol{G}_A, A = 1, 2$ in the reference configuration. We augment them with the third direction $\boldsymbol{G}_3 := \boldsymbol{G}_1 \times \boldsymbol{G}_2 / |\boldsymbol{G}_1 \times \boldsymbol{G}_2|$. We consider $\boldsymbol{G}_A$ a basis of the curvilinear system of coordinates corresponding to some parametrization $\boldsymbol{X} = \boldsymbol{X}(\xi^A), A = 1, 2, 3$, i.e. $\boldsymbol{G}_A = \partial \boldsymbol{X} / \partial \xi^A$. We consider also the basis $\boldsymbol{G}^A$ of the adjoint space (of linear functionals) which is dual to $\boldsymbol{G}_A$:

$$\langle \boldsymbol{G}^A, \boldsymbol{G}_B \rangle = \delta^A_B. \tag{2.1}$$

We use convective spatial coordinates, i.e. the spatial basis vectors are generated by the parametrization $\boldsymbol{x} = \boldsymbol{x}(\zeta^a)$, $\boldsymbol{g}_a = \partial \boldsymbol{x} / \partial \zeta^a$ for wich $\boldsymbol{\zeta} = \boldsymbol{\xi}$. It is known that the deformation gradient $\boldsymbol{F}$, its adjoint $\boldsymbol{F}^*$ and their inverses $\boldsymbol{F}^{-1}$ and $\boldsymbol{F}^{-*}$ take the form:

$$\begin{aligned} \boldsymbol{F} &= \delta^a_A \boldsymbol{g}_a \otimes \boldsymbol{G}^A, \quad \boldsymbol{F}^{-1} = \delta^A_a \boldsymbol{G}_A \otimes \boldsymbol{g}^a, \\ \boldsymbol{F}^* &= \delta^a_A \boldsymbol{G}^A \otimes \boldsymbol{g}_a, \quad \boldsymbol{F}^{-*} = \delta^A_a \boldsymbol{g}^a \otimes \boldsymbol{G}_A, \end{aligned} \tag{2.2}$$

where $\boldsymbol{g}^a$ denotes the adjoint basis dual to $\boldsymbol{g}_a$, i.e. satisfying the condition $\langle \boldsymbol{g}^a, \boldsymbol{g}_b \rangle = \delta^a_b$. We also introduce the material and spatial metric tensors:

$$\boldsymbol{G} = G_{AB} \boldsymbol{G}^A \otimes \boldsymbol{G}^B \quad \text{and} \quad \boldsymbol{g} = g_{ab} \boldsymbol{g}^a \otimes \boldsymbol{g}^b, \tag{2.3}$$

where $G_{AB} := \boldsymbol{G}_A \cdot \boldsymbol{G}_B$ and $g_{ab} := \boldsymbol{g}_a \cdot \boldsymbol{g}_b$. The right Cauchy-Green deformation tensor takes the form:

$$\boldsymbol{C} = \boldsymbol{F}^* \boldsymbol{g} \boldsymbol{F}, \quad \boldsymbol{C} = \delta^a_A \delta^b_B g_{ab} \boldsymbol{G}^A \otimes \boldsymbol{G}^B. \tag{2.4}$$

We also introduce the structural tensors corresponding to the preferred directions of fibres:

$$\boldsymbol{A}_F := \boldsymbol{G}_F \otimes \boldsymbol{G}_F \quad \text{(no sum)}, \quad F = 1, 2. \tag{2.5}$$

We can easily verify that the stretches λ_F of the preferred directions can be expressed as follows:

$$\lambda_F = \langle \boldsymbol{C}, \boldsymbol{A}_F \rangle^{1/2}. \tag{2.6}$$

We define the stretchless part $\overleftrightarrow{\boldsymbol{C}}$ of the right Cauchy-Green tensor (analogous to the unimodular $\bar{\boldsymbol{C}}$ for incompresible analysis):

$$\overleftrightarrow{\boldsymbol{C}} := \boldsymbol{C} - \sum_F (\lambda_F^2 - 1)\boldsymbol{A}^F, \tag{2.7}$$

where $\boldsymbol{A}^F := \boldsymbol{G}^F \otimes \boldsymbol{G}^F$. We also define the projection operator $\overleftrightarrow{\mathbb{P}}$ which extracts the tensionless part of the stress:

$$\overleftrightarrow{\mathbb{P}} := \mathbb{I} - \sum_F \boldsymbol{A}_F \otimes \boldsymbol{A}^F. \tag{2.8}$$

With these notions we propose the ansatz for the strain energy function $\Psi = \Psi(\boldsymbol{C}; \boldsymbol{A}_1, \boldsymbol{A}_2)$ in the form taking into account that $\boldsymbol{C}$ is expressed by the dependent on deformation tensor $\overleftrightarrow{\boldsymbol{C}}$ and separately approximated stretches $\tilde{\lambda}_F$:

$$\tilde{\Psi}(\overleftrightarrow{\boldsymbol{C}}, \tilde{\lambda}_1, \tilde{\lambda}_2; \boldsymbol{A}_1, \boldsymbol{A}_2) = \Psi(\boldsymbol{C}; \boldsymbol{A}_1, \boldsymbol{A}_2) \tag{2.9}$$

in which we shall enforce decoupling between the dependence on $\overleftrightarrow{\boldsymbol{C}}$ and $\tilde{\lambda}_F$:

$$\frac{\partial^2 \Psi}{\partial \overleftrightarrow{\boldsymbol{C}}\, \partial \tilde{\lambda}_F} = 0. \tag{2.10}$$

Selecting separate approximation of stretches $\tilde{\lambda}_F$ suggests assuming the following augmented decoupled strain energy ansatz:

$$\begin{cases} \Psi = \tilde{\Psi}(\overleftrightarrow{\boldsymbol{C}}, \tilde{\lambda}_1, \tilde{\lambda}_2; \boldsymbol{A}_1, \boldsymbol{A}_2) - \sum_F \tilde{\rho}_F[\tilde{\lambda}_F - \lambda_F(\boldsymbol{C})], \\ \tilde{\Psi} = \sum_F \Psi_{\|F}(\tilde{\lambda}_F) + \overleftrightarrow{\Psi}(\overleftrightarrow{\boldsymbol{C}}, \boldsymbol{A}_1, \boldsymbol{A}_2). \end{cases} \tag{2.11}$$

The assumptions above and the Clausius-Plank inequality lead to the following constitutive equations for the 2nd Piola-Kirchhoff stress:

$$\begin{cases} \boldsymbol{S} = \sum_F \tilde{\rho}_F \lambda_F^{-1} \boldsymbol{A}_F + \overleftrightarrow{\boldsymbol{S}}, \\ \overleftrightarrow{\boldsymbol{S}} = \overleftrightarrow{\mathbb{P}} \left[2 \dfrac{\partial \overleftrightarrow{\Psi}}{\partial \overleftrightarrow{\boldsymbol{C}}} \right], \\ \tilde{\rho}_F = \Psi'_{\|F}(\tilde{\lambda}_F), \\ \tilde{\lambda}_F = \lambda(\boldsymbol{C})_F. \end{cases} \tag{2.12}$$

A particular selection of the strain energy functions $\overleftrightarrow{\Psi}$ and $\Psi_{\|F}$ follows the suggestions of Balzani et al. [4] and Simo and Pister [5]:

$$\begin{cases} \frac{2}{\mu} \overleftrightarrow{\Psi} = \bar{\Lambda}[\ln \overleftrightarrow{J}]^2 - 2\ln(\overleftrightarrow{J}) + (\overleftrightarrow{I}_1 - 3) + \sum_F \bar{\Phi}(\overleftrightarrow{K}_{1,F} - 1)^2, \\ \frac{2}{\mu} \Psi_{\|F}(\tilde{\lambda}_F) = \bar{\Gamma}\, (\tilde{\lambda}_F^2 - 1)^2, \end{cases} \tag{2.13}$$

where μ is the shear modulus, $\overleftrightarrow{I}_1, \overleftrightarrow{I}_2$ are the 1st and 2nd invariants of $\overleftrightarrow{\boldsymbol{C}}$, $\overleftrightarrow{K}_{1,F} = \overleftrightarrow{I}_a - \overleftrightarrow{I}_1 + \overleftrightarrow{I}_2, a = 2^F + 3, F = 1, 2$ while $\overleftrightarrow{I}_5, \overleftrightarrow{I}_7$ are the joint invariants of $\overleftrightarrow{\boldsymbol{C}}$ and $\boldsymbol{A}_F$: $\overleftrightarrow{I}_a = \langle \overleftrightarrow{\boldsymbol{C}}^2, \boldsymbol{A}_F \rangle$. Parameters $\bar{\Lambda}, \bar{\Phi}, \bar{\Gamma}$ are the nondimentional material characteristics. The mixed formulation for $(\boldsymbol{u}, \tilde{\rho}_F, \tilde{\lambda}_F)$ involves the principle of virtual work (expressing equilibrium), the identification $\tilde{\lambda}_F = \lambda_F(\boldsymbol{C})$ and the constitutive relation for $\tilde{\rho}_F$, and it takes the form: find $(\boldsymbol{u}, \tilde{\rho}_F, \tilde{\lambda}_F) \in (V + \boldsymbol{u}_0) \times Q^4$ such that:

$$\begin{cases} \int_\Omega \langle D_u \boldsymbol{E}(\boldsymbol{u})[\delta \boldsymbol{u}], \boldsymbol{S} \rangle dV = \int_\Omega \langle \boldsymbol{G}\delta\boldsymbol{u}, \bar{\boldsymbol{B}} \rangle dV + \int_{\Gamma_N} \langle \boldsymbol{G}\delta\boldsymbol{u}, \bar{\boldsymbol{P}} \rangle dA, \\ \int_\Omega \delta\tilde{\rho}_F \{\lambda_F(\boldsymbol{C}) - \tilde{\lambda}_F\} dV = 0, \\ \int_\Omega \delta\tilde{\lambda}_F \{\Psi'_{\|F}(\tilde{\lambda}_F) - \tilde{\rho}_F\} dV = 0, \end{cases} \tag{2.14}$$

for all $\delta\boldsymbol{u} \in V, \delta\tilde{\lambda}_F, \delta\tilde{\rho}_F \in Q$, $V = \{\boldsymbol{v} \in H^1(\Omega) : \boldsymbol{v} = 0, \text{ on } \Gamma_D\}, Q = L^2(\Omega)$. In (2.14) $\bar{\boldsymbol{B}}$ denotes the volume forces, $\bar{\boldsymbol{P}}$ and $\boldsymbol{u}_0$ are the Neumann and Dirichlet data on Γ_N and Γ_D. In addition $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$ and $D_u\boldsymbol{E}(\boldsymbol{u})[\delta\boldsymbol{u}] = \frac{1}{2}(\boldsymbol{F}^*\boldsymbol{\nabla}\delta\boldsymbol{u} + \boldsymbol{\nabla}^*\delta\boldsymbol{u}\boldsymbol{F})$. The FE approximation of (2.14) results in a system of nonlinear equations which is solved using the Newton-Raphson algorithm applied to linearization of (2.14).

3. A numerical example

We solved with an h-adaptive FEM presurization of a tube of the height $h = 10$ and the internal and external radii $r_1 = 3.317$ and $r_2 = 4.057$. The tube is reinforced with two families of fibres which constitute the spiral lines inclined by the angle of 30^0 to the horizontal plane. The material data are as follows: $\mu = 1$, $\bar{\Lambda} = 1$, $\bar{\Phi} = 1$, $\bar{\Gamma} = 10$. The loading internal pressure $p = 2.0$. Figure 1 presents the contour map of the Kirchhoff stress $\sigma_{\phi\phi}$ on the deformed configuration after performing 4 steps of refinement on the initial mesh of $4 \times 4 \times 1$ elements.

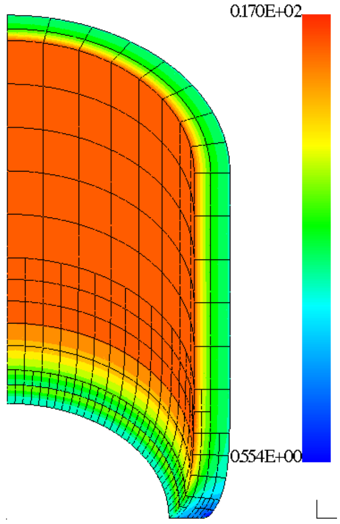

Figure 1: Pressurization of a tube on a $p = 2$ mesh. Contour map of $\sigma_{\phi\phi}$ (circumferential stress) on a deformed configuration

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