Three-dimensional analysis of free vibrations in homogeneous and laminated plates using two-dimensional numerical model

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Abstract

In the paper the problem of free vibrations of rectangular plates is approached. Both, homogeneous plates and multi-layered laminates are taken into account. In the analysis the method called FEM23 is used. The name derives from the fact that three-dimensional (3D) finite element results are obtained on the basis of two-dimensional (2D) mesh. Within the method, the 3D approximation is constructed as a combination of 2D in-plane approximation with 1D transverse approximation. The in-plane approximation is constructed using 2D planar finite element mesh while for 1D transverse approximation the Lagrange polynomials are applied. The orders of in-plane and transverse approximations are independent of each other. The full 3D mathematical model of free vibrations is calculated on in-plane 2D mesh. The smart postprocessing tailored to FEM23 is applied for 3D visualisation of the solution. The results of numerical calculations for rectangular plates with various boundary conditions are presented. Solutions obtained using FEM23 are compared with the analytical results or with the ones computed using well established FEM package.

Keywords: finite element method, free vibrations, rectangular plates

1. Introduction

The problem of 3D numerical modelling of free vibrations of rectangular homogeneous and laminated plates is addressed in the paper. The natural frequencies and the forms of vibrations are aimed to be determined in the analysis. Laminated composites plates are frequently used in engineering applications due to their properties such as high strength and durability in respect to weight. Thus, the numerical modelling of laminated plates is found in the mainstream of scientific research, e.g. [1].

The two-dimensional (2D) numerical model is formulated for a robust full three-dimensional (3D) numerical analysis of the eigenvalue problem. The method called FEM23 (3D finite elements method on 2D mesh) [2, 3] is used. It means that only 2D discretisation for the whole structure is applied, but full 3D results are obtained. FEM23 is suitable for the analysis of both homogeneous and laminated plates. It is assumed in this work that the laminated plate consists of a set of homogeneous layers. The FEM23 is based on combination of 2D in-plane approximation with 1D transverse approximation. The orders of in-plane and transverse approximations are independent of each other. The in-plane approximation order results from the order of finite elements (FE) in a 2D mesh, while in the transverse direction the approximation may change from the 1st up to the 4th order for each layer, depending on the layer thickness. The method is completed with special post-processing in which full 3D results for laminated plate may be visualised. In the standard 3D FEM approach the 3D mesh has to be used. In a situation when laminated structure is analysed, 3D meshing has to be adjusted to the thickness of layers. In FEM23 only 2D mesh is used and the structure may consists of thick as well as with very thin layers. The order of transverse approximation can be adjusted to the thickness of layers, for example 1st order for thin layers and up to 4th order for thick layers whereas the in-plane order is set independently. FEM23 is suitable to model thin and thick plates, in contrary to standard 2D plates models where various hypothesis are used depending on the plate thickness, see [4].

For the sake of clarity the FEM23 is presented for homogeneous plate which, afterwards, may be easily extended to multi-layered plate.

2. Numerical model of free vibrations

If the structure load (value, direction, point of application) changes in time or has impact type (suddenly applied/removed load) the dynamic analysis of the structure is required. The variational form of the equilibrium equation for the considered problem, with appropriate boundary conditions, can be written as

$$\int_V \boldsymbol{\varepsilon}(\mathbf{V}) : \mathbf{E} : \boldsymbol{\varepsilon}(\mathbf{U})\,\mathrm{d}V - \int_V \rho f^2\,\mathbf{V}\cdot\mathbf{U}\,\mathrm{d}V = 0 \tag{1}$$

where $\mathbf{E}$ is the Hooke's elasticity tensor, f is the frequency of vibrations, $\mathbf{U}$ is the function of vibration amplitudes and $\mathbf{V}$ is the test function. The small strains are assumed so in eq. (1) the Cauchy strain is used

$$\boldsymbol{\varepsilon}(\mathbf{U}) = \frac{1}{2}\Big(\boldsymbol{\nabla}\mathbf{U} + (\boldsymbol{\nabla}\mathbf{U})^{\mathrm{T}}\Big) \tag{2}$$

The spatial approximation of $\mathbf{U}$ is presented as combination of transverse and in-plane approximations

$$\mathbf{U}(x,y,z) = \sum_i^n N_i(z)\boldsymbol{\Phi}(x,y)\check{\mathbf{U}}^i = \boldsymbol{\Psi}(x,y,z)\check{\mathbf{U}} \tag{3}$$

where N_i are the 1D Lagrange polynomials, $\mathbf{\Phi}$ is the approximation matrix based on 2D in-plane finite element mesh, $\mathbf{\Psi}$ is the 3D approximation matrix and $\check{\mathbf{U}}$ is the vector of degrees of freedom.

Following the procedure described in detail in [3] the volume integral are composed of combinations of 2D in-plane integral with 1D transverse integral. In the transverse direction the explicit Gauss integration is applied. Finally, eq. (1) leads to discrete eigenvalue problem that has the following matrix form

$$\left(\mathbf{K}_1+\mathbf{K}_2+\mathbf{K}_3+\mathbf{K}_4-f^2\mathbf{M}\right)\check{\mathbf{U}}=\mathbf{0} \tag{4}$$

where the matrices $\mathbf{K}_i$ and $\mathbf{M}$ are expressed with the help of 2D in-plane integrals. For example the definitions of $\mathbf{K}_2$ and $\mathbf{M}$ are as follows

$$\mathbf{K}_2=\int_{S^m}\sum_k^g w_k\left(\mathbf{\Psi},_z^{\mathrm{T}}\cdot\mathbf{E}_2:\bar{\mathbf{\nabla}}\mathbf{\Psi}\right)\Big|_{z_k}\mathrm{d}S \tag{5}$$

$$\mathbf{M}=\int_{S^m}\rho\sum_k^g w_k\left(\mathbf{\Psi}^{\mathrm{T}}\cdot\mathbf{\Psi}\right)\Big|_{z_k}\mathrm{d}S \tag{6}$$

where $\mathbf{E}_2$ is the appropriate tensors based on the Hooke's elasticity tensor, $\bar{\mathbf{\nabla}}$ is the in-plane gradient operator, z_k and w_k are the appropriate point and weight of Gaussian quadrature, respectively.

3. Examples

To show the efficiency and accuracy of the method the results of calculations for rectangular plates with various boundary conditions are presented. Solutions obtained using FEM23 are compared with the analytical results, e.g. [5], or with the ones computed using commercial FEM package. Two examples are presented here where the simply supported thin plate and in the following the clamped laminated glass are considered.

3.1. *Free vibrations of the simply supported plate*

The thin square plate is considered. The results are compared with analytical solution for thin plate model [4]. The following geometrical and material data are adopted: side length $a = 1.2$ m, thickness $h = 0.0151$ m, Young's modulus $E = 1.47\,10^8$ kN/m^2, Poisson's ratio $\nu = 0.3$, material density$\rho = 10^4$ kg/m^3.

The analytical solution is obtained solving the following four-order differential equation

$$\nabla^2\nabla^2 W(x,y)-\frac{\mu}{D^m}\,\omega^2\,W(x,y)=0 \tag{7}$$

where D^m is the bending stiffness, $\mu = \rho\,h$, $W(x,y)$ is the deflection amplitude and $\omega = 2\Pi f$ is angular frequency of vibrations. In this case the analytical solution can be obtained using Navier's method and is expressed by the function

$$W(x,y)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}W_{ij}\,\sin\frac{i\,\pi\,x}{a}\,\sin\frac{j\,\pi\,y}{a} \tag{8}$$

that satisfies the appropriate mixed boundary conditions. The obtained analytical values of ω are as follows

$$\omega_{(i,j)}=\frac{\pi^2}{a^2}\,\left(i^2+j^2\right)\,\sqrt{\frac{D^m}{\mu}} \tag{9}$$

Finally, the results of analytical calculations reads: $\omega_{(1,1)} = 240.16$ rad/s, $\omega_{(1,2)} = 600.40$ rad/s, $\omega_{(2,2)} = 960.64$ rad/s. Similar results have been obtained using FEM23 method: $\omega_{(1,1)} = 239.5$ rad/s, $\omega_{(1,2)} = 599.5$ rad/s, $\omega_{(2,2)} = 957.3$ rad/s.

3.2. *Free vibrations of laminated glass*

The rectangular 2×1 [m] laminated glass (LG) plate is analysed in this example. The LG consists of two 5 [mm] glass panes bonded by 0.38 [mm] layer of PVB (polyvinyl butyral). The plate is clamped on the one short side and supported on the opposite side. The material parameters for LG are shown in Tab. 1.

Table 1: Material parameters for glass and PVB.

Property	Material	
	Glass	PVB
E [GPa]	64.3	3.73×10^{-3}
ν	0.23	0.45
ρ [kg/m^3]	2300	1100

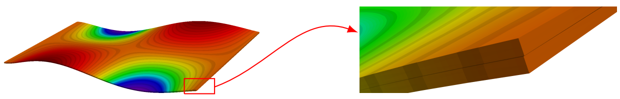

Figure 1: Laminated glass plate – the fourth mode of vibrations.

The computed values of the natural frequencies reads: $\omega_{(1,1)} = 63.78$ rad/s, $\omega_{(1,2)} = 112.28$ rad/s, $\omega_{(2,2)} = 171.79$ rad/s. For analogous monolithic (glass) plate the obtained values are: $\omega_{(1,1)} = 70.92$ rad/s, $\omega_{(1,2)} = 139.95$ rad/s, $\omega_{(2,2)} = 212.62$ rad/s. In the Fig. 1 the fourth mode of vibrations of the LG is shown where the one vertex is focused. In spite of the fact that the PVB layer is very thin the FEM23 is able to perform full 3D analysis of such a structure.

4. Conclusions

In this work full 3D free vibrations analysis has been performed for homogeneous and laminated plates using FEM23. In this method only 2D in-plane mesh is needed for 3D modelling. In a case of homogeneous thin plate a great agreement has been achieved between numerical modelling and analytical solution. The effectiveness and flexibility of the FEM23 have been proved by the successful analysis of the complex laminated structure. In a case of laminated glass the inner bonding PVB layer is extremely thin since it thickness is over 2500 times smaller in relation to characteristic plate dimension. It has been shown that FEM23 is able to perform the 3D free-vibrations analysis for such a structure.

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