Enhanced heat flux prediction using X-FEM and a recovery procedure for discontinuous problems

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Abstract

In the paper, the extended finite element method (XFEM) is combined with a heat flux recovery procedure in the analysis of a steady-state heat conduction problem with discontinuous terms. Computationally efficient low-order finite elements provided good convergence are used. The combination of the XFEM with a recovery procedure allows for quadratic convergence rates in heat flux solutions i.e. as the same order as temperature solution. The discontinuity is modelled independently of the finite element mesh using a novel extended enrichment functions in the finite element approximation. The results show improved heat flux prediction locally for the interface element and globally for the entire domain.

Keywords: weak discontinuity, XFEM, heat flux recovery, heat conduction

1. Introduction

The aim of the paper is to reach enhanced heat flux predictions, as part of the solution of the Poisson equation which represents the mathematical model of the problem. During the last decades much effort was paid to explore a stress improvement procedures for solid and structures and to establish solution error estimates, [3], [1]. Recently an effective technique has been proposed in [4]. In the case of discontinuous problems the XFEM approximation can recover the discontinuity of the solution locally using enriched approximation, [2]. The XFEM solution introduce the discontinuity through nodal enrichment function, and controls it by additional degrees of freedom. This allows to make the finite element mesh independent of discontinuity location. The quality of the solution depends mainly on the assumed enrichment basis functions. In order to achieve better approximation, the new enrichment functions are proposed.

If an effective procedure to improve the solution is established, a coarse mesh with low-order elements can be used in the finite element analysis. In a study, we asses the effectiveness of the proposed approach using L_2 and energy norms.

2. Governing equations

Let us consider a domain Ω with boundary Γ divided into the sub-domains Ω_S and Ω_L. The sub-domains are separated from each other by the interface Γ_I. The boundary Γ is composed of the sets Γ_D and Γ_N such that $\Gamma = \Gamma_D \cup \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. The normal vector $\mathbf{n}$ on the external boundary Γ and the outward normal vector $\mathbf{n}_I$, on the interface Γ_I are defined.

The evaluation of temperature $T(\mathbf{x})$ in Ω is governed by the Poisson equation

$$\bigtriangledown \mathbf{q}(\mathbf{x}) + f(\mathbf{x}) = 0 \quad \text{in} \quad \Omega \tag{1}$$

where

$$\mathbf{q}(\mathbf{x}) = -k \bigtriangledown T(\mathbf{x}) \tag{2}$$

and k is the thermal conductivity and f denotes a heat source.

The boundary conditions define the temperature at the boundary Γ_D

$$T(\mathbf{x}) = T_D(\mathbf{x}) \quad \text{at} \quad \Gamma_D \tag{3}$$

and the heat flux at Γ_N

$$-k \bigtriangledown T(\mathbf{x}) \cdot \mathbf{n} = q_S(\mathbf{x}) \quad \text{at} \quad \Gamma_N. \tag{4}$$

It is assumed that the coefficient k and the function f are discontinuous across the interface boundary Γ_I.

The finite element equation is obtained from the weak form of the problem

$$\mathbf{K}^{(e)}\mathbf{u}^{(e)} = \mathbf{f}^{(e)} - \mathbf{q}_S^{(e)} \tag{5}$$

where $\mathbf{u}^{(e)}$ is a vector of nodal unknowns, and $\mathbf{K}^{(e)}$, $\mathbf{f}^{(e)}$ and $\mathbf{q}_S^{(e)}$ are the stiffness matrix, load vector and externally applied heat flux, respectively. Their global counterparts are computed through the usual assembly procedure.

3. XFEM and recovery procedure

The finite element method uses continuous approximation within individual element. Thus, it can only be applied to solve the discontinuous problems by aligning the mesh with discontinuity. The XFEM is suitable for describing discontinuities in the solution fields independent of the finite element mesh. It is essential, however, to locally apply special approximation functions. The enrichment area is contained to the vicinity of a discontinuity; as a result, the size of the problem remains relatively unchanged.

A solution characteristic of the problem is introduced by adding the enrichment term $T(\mathbf{x}, \mathbf{x}_I)_E$ to the standard finite element approximation $T(\mathbf{x})_C$

$$T_h(\mathbf{x}) = T(\mathbf{x})_C + T(\mathbf{x}, \mathbf{x}_I)_E \tag{6}$$

where

$$T(\mathbf{x})_C = \sum_{j \in I} N_j(\mathbf{x}) T_j \tag{7}$$

The enrichment term $T(\mathbf{x}, \mathbf{x}_I)_E$ combines the enrichment functions $\Psi^\alpha(\mathbf{x}, \mathbf{x}_I)$ with a partition of unity (PU) functions $\mathbf{N}(\mathbf{x})$ (usually element shape functions)

$$T(\mathbf{x}, \mathbf{x}_I)_E = \sum_{j \in J} \sum_{\alpha=1}^{m} N_j(\mathbf{x}) \Psi^\alpha(\mathbf{x}, \mathbf{x}_I) a_j^\alpha \tag{8}$$

where J is the set of nodes enriched by $\Psi^\alpha(\mathbf{x}, \mathbf{x}_I)$, a_j^α are the additional degrees of freedom, I is the set of all nodes and m is the number of enrichment functions and $\mathbf{x}_I$ denotes that a term depends on the interface position.

Then, we use equation (5) to obtain unknown coefficients of approximation $\mathbf{T}$ and $\mathbf{a}$ gathered in vector $\mathbf{u}$. Finally, the heat flux in an element is calculated using equations (6-8) and (2).

Now, we briefly present the recovery procedure proposed in [4]. The procedure is applied to all elements with continuous approximation. It is assumed that the approximation in enriched elements allows for quadratic convergence in heat flux.

The temperature pattern within each element with continuous approximation is represented by equation (7). With this assumption, the heat flux $\mathbf{q}_h^{(e)}(\mathbf{x})$ of element e follows from equation (2). We refer to this heat flux as the directly calculated finite element heat flux. It is well known, that for the low order elements (2-node element in 1D, 3-node element in 2D, etc.) the quality of calculated heat flux is poor as compared to the calculated temperature field.

In the presented formulation, the accurate heat flux prediction is obtained using a mixed interpolation approach. In this approach, the Lagrange multiplier technique is utilised in order to apply physical relationship (2) over the element volumes. The additional solution variable are then the heat flux coefficients $\mathbf{q}^{(e)}$ and Lagrange multipliers $\boldsymbol{\lambda}^{(e)}$, which are defined by internal degrees of freedom. They are related only to the considered element e. An important feature of the mixed formulation is that the temperature problem is decoupled from the calculation of the enhanced heat flux $\mathbf{q}^{(e)}(\mathbf{x})$.

In order to deliver improved heat flux prediction a richer space for $\mathbf{q}^{(e)}(\mathbf{x})$ must be assumed than that assumed for $\mathbf{q}_h^{(e)}(\mathbf{x})$. The interpolating functions used in the calculation are those proposed in [4].

4. 1D solution

In this section, we test the procedure for 1D discontinuous problem in $\Omega = \langle 0, 1 \rangle$ and arbitrary loading and material properties. The boundary Γ_I is defined by a single point $x_I = 0.6667$. It is assumed that the cross section area of the 1D structure is constant. Let us consider the equation (1) with the following terms

$$k = \begin{cases} k_L = 0.1 & \text{for} \quad x < x_I \\ k_S = 1 & \text{for} \quad x > x_I \end{cases} \tag{9}$$

and the homogeneous Dirichlet boundary conditions

$$T(x = 0) = T(x = 1) = 0 \tag{10}$$

The forcing term is assumed to be a smooth function in the form

$$f(x) = 10 \sin\left(\frac{2\pi x}{x_I}\right). \tag{11}$$

The finite elements with discontinuous approximation are enriched with the set of two functions. The common abs-enrichment with shifted-basis approximation is combined with a tested quadratic function

$$\begin{aligned} \Psi_j^1(x) &= |x - x_I| - |x_j - x_I| \\ \Psi_j^2(x) &= |x - x_I| g(x) - |x_j - x_I| \end{aligned} \tag{12}$$

where

$$g(x) = \begin{cases} N_1(x) & \text{if} \quad x < x_I \\ N_2(x) & \text{if} \quad x > x_I \end{cases} \tag{13}$$

It is assumed that the considered enrichment functions allows for a better prediction of the heat flux in reproducing elements in comparison to the standard abs-enrichment.

Figure 1 shows the heat flux convergence curves for the 1D problem measured in the H^1 norm and the convergence of the temperature results measured in the sense of L^2 norm.

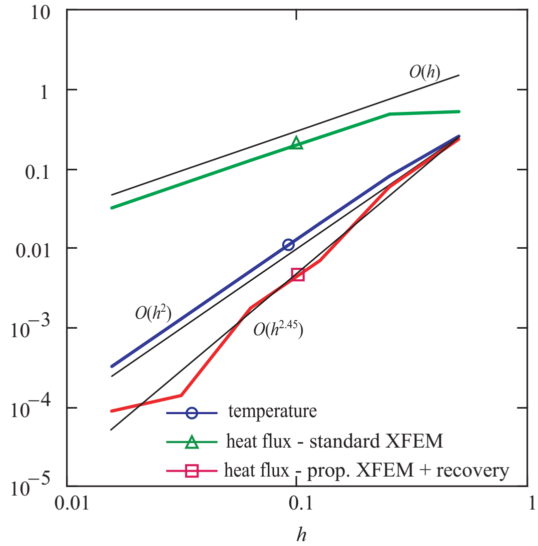

Figure 1: Convergence curves for the 1D problem.

Considering the results we see, that the rate of convergence for the temperature $T(x)$ is $O(h^2)$ whereas the enhanced heat flux $q(x)$ converges even faster, reaching $O(h^{2.45})$ in the H^1 norm. Further, the solution for the heat flux is much more accurate then directly given by temperature.

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