Comparison of two types of fractional operators including both the left and right fractional derivatives

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Abstract

In the present paper we study two types of fractional operators including both the left and right fractional Caputo derivatives of order $\alpha \in (0, 1)$. The first appears in the fractional Euler-Lagrange mechanics. The second is from the fractional continua. Next, the selected functions are differentiated by using the mentioned differential operators of fractional order. The obtained results are shown in the given figures.

Keywords: Caputo derivatives, fractional operators, fractional continua, fractional Euler-Lagrange mechanics

1. Introduction

Fractional operators, both integral and differential, have attracted the attention of researchers from different fields of sciences (like physics, bioengineering, mechanics, control theory or economy [2, 4, 8]) over the last decade. The mean reason of this fact is that the fractional derivatives are nonlocal operators in contrast to integer order derivatives.

It has been demonstrated that the fractional order representation provides a more realistic behavior of complex systems appearing in various fields of science and engineering. One can find many different approaches in formulating fractional models. It can be the result of applying various types of fractional derivatives.

The most common derivatives are the Caputo, Riemann-Liouville or Grunwald-Letnikov operators [5, 6].

We can indicate many other definitions, however we focus our attention on differential operators which are a kind of composition of the left and right fractional derivatives. The first one appears in the fractional Euler-Lagrange mechanics [1, 3] and the second one in the fractional continua [7].

2. Fractional preliminaries

Let us start from definitions of fractional derivatives [5]. First, we recall the left and right Caputo fractional derivatives of order $\alpha \in (0, 1)$, respectively

$$ {}^{C}D^{\alpha}_{a+}\, y(x) := I^{1-\alpha}_{a+} y'(x) \tag{1} $$

$$ {}^{C}D^{\alpha}_{b-}\, y(x) := -I^{1-\alpha}_{b-} y'(x), \tag{2} $$

where y' denotes the first order derivative and operators I^{α}_{a+} and I^{α}_{b-} are respectively the left and the right Riemann-Liouville fractional integrals of order $\alpha > 0$ defined by

$$ I^{\alpha}_{a+} y(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{y(s)}{(x-s)^{1-\alpha}} ds \quad \text{for } x > a \tag{3} $$

$$ I^{\alpha}_{b-} y(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{y(s)}{(s-x)^{1-\alpha}} ds \quad \text{for } x < b \tag{4} $$

where Γ is the well known Gamma function. If $\alpha = 1$ then ${}^{C}D^{1}_{a+}\, y = y'$ and ${}^{C}D^{1}_{b-}\, y = -y'$.

3. Two types of fractional differential operators

In this section we focus our attention on two types of fractional operators which include both the left and right Caputo fractional derivatives.

The first one appears when the variational approach is applied in the fractional calculus [1, 3]. This type of operator is the composition of the left and right fractional derivatives

$$ \begin{aligned} {}^{RL}D^{\alpha} y(x) &= {}^{C}D^{\alpha}_{b-}\, {}^{C}D^{\alpha}_{a+} y(x) \\ &= \frac{-1}{\Gamma^2(1-\alpha)} \int_x^b \frac{1}{(s-x)^{\alpha}} \frac{d}{ds} \left(\int_a^s \frac{y'(\tau)}{(s-\tau)^{\alpha}} d\tau \right) ds \end{aligned} \tag{5} $$

The second approach comes from the application of the fractional calculus in continuum mechanics [7], and the used fractional model includes the following differential operator

$$ \begin{aligned} {}^{RC}D^{\alpha} y(x) &= \frac{\Gamma(2-\alpha)}{2} \frac{d}{dx} \left({}^{C}D^{\alpha}_{a+} y(x) - {}^{C}D^{\alpha}_{b-} y(x) \right) \\ &= \frac{1-\alpha}{2} \left(\frac{d}{dx} \int_a^x \frac{y'(s)}{(x-s)^{\alpha}} ds + \frac{d}{dx} \int_x^b \frac{y'(s)}{(s-x)^{\alpha}} ds \right) \end{aligned} \tag{6} $$

4. Comparison of fractional derivatives

In this section we compare the fractional derivatives described in the previous section. We do this by testing how the studied operators act on the selected functions. We consider two linear functions $y(x) = x$ and $y(x) = 1 - x$. We differentiate the indicated functions by using fractional derivatives (5), and (6). Next, the obtained results are presented in Figures 1 - 4.

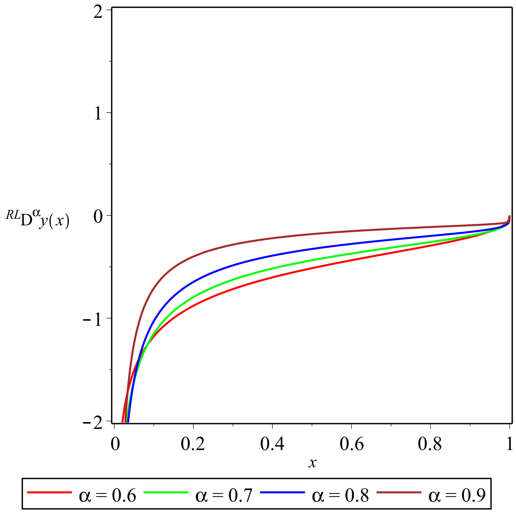

Figure 1: Plot of the fractional derivative (5) of the function $y(x) = x$ for various orders α

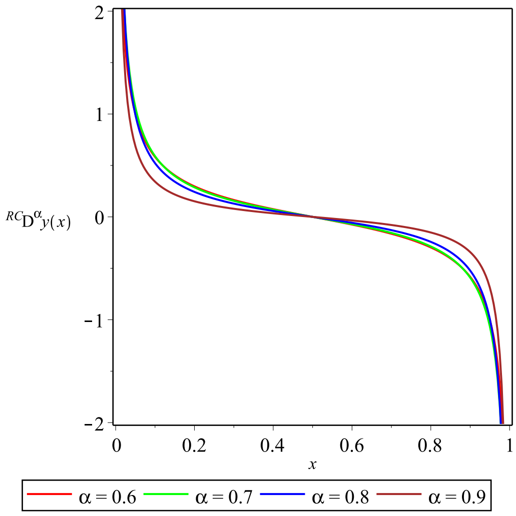

Figure 2: Plot of the fractional derivative (6) of the function $y(x) = x$ for various orders α

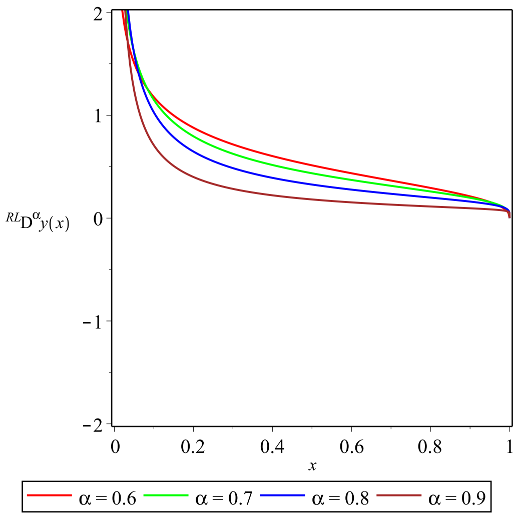

Figure 3: Plot of the fractional derivative (5) of the function $y(x) = 1 - x$ for various orders α

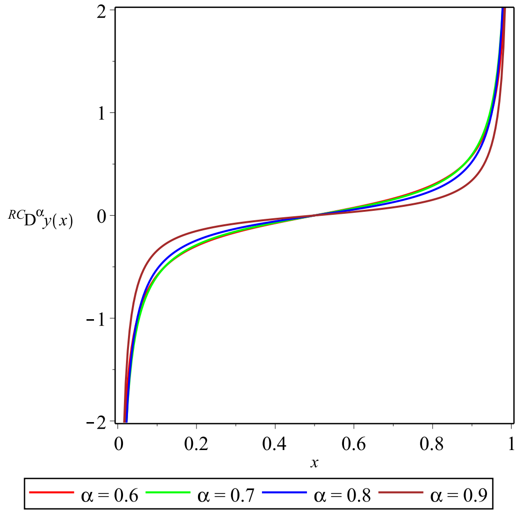

Figure 4: Plot of the fractional derivative (6) of the function $y(x) = 1 - x$ for various orders α

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