On selected aspects of fractional plasticity

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Abstract

Generalization of Perzyna's type model of viscoplasticity utilizing fractional operator is considered, called *fractionalviscoplasticity*. The main objective of this research is to present selected properties (eg. rate effects, anisotropy of plastic flow) of amodel where Riesz- Caputo derivative was introduced instead of integer order derivative. This new approach allows to abandon theadditional potential assumption required to achieve a non-associative flow. Model discussed in this paper introduces new materialconstants, for instance the order of the fractional flow or the limits of the flow anisotropy. The relation between these parameters andthe plastic deformation was illustrated in numerical examples.

Keywords: fractional calculus, Perzynaviscoplasticity, fractional viscoplasticity, fractional mechanics

1. Introduction

Material modelling is essential for reflecting properties andbehaviour of physical objects. Theory of fractional plasticity introducesrule of non-associated flow which increases accuracy ofmodels of metals, granular materials, concrete, ceramics andcomposites [9]. This phenomena poses a challenge for classicaltheory that may be solved with fractional flow rule [4], [5], [6],[7], [8]. The Caputo derivative, discussed in this paper, is definedover an interval in adversity to the classical derivative of integerorder [1]. This allow to govern and encode properties of a materialfrom a nonlocal neighbourhood in stress space which mayyield better results in modelling of continuous body.

2. Fractional viscoplasticity

2.1. Perzyna type viscoplasticity

Concept of viscoplasticity was first presented by Perzyna [2]as a generalization of one-dimensional constitutive equation forrate-sensitive materials. Rate of viscoplastic strain can be formulatedas

$$\dot{\varepsilon}^p = \Lambda \boldsymbol{p}, \tag{1}$$

where Λ is a scalar multiplier and $\boldsymbol{p}$ denotes the second-order tensor describing the direction of the plastic flow. In the classical approach it is assumed that the direction of plastic flow is normal to the yield surface, hence p is equal to

$$\boldsymbol{p} = \frac{\partial F}{\partial \sigma}\left(\left\|\frac{\partial F}{\partial \sigma}\right\|\right)^{-1}, \tag{2}$$

where F is a plastic potential and σ is a second-order stress tensor. Intensity of the viscoplastic flow in enclosed in the definition of multiplier

$$\Lambda = \gamma\langle\Phi(F)\rangle, \tag{3}$$

where γ is a viscosity parameter, <.> denotes the Macaulay brackets and $\Phi(F)$ is a overstress function of F.

2.2. Fractional formulation of plastic flow

It should be noted that viscoplastic flow is a deviatoric type,therefore it does not consider the volume change of the material.In order to control the volume in plastic range, once the yield criterionis fulfilled, a fractional differentiation of a potential can beapplied, namely

$$\boldsymbol{p} = D^\alpha F \|D^\alpha F\|^{-1}, \tag{4}$$

where D^α denotes the fractional differential operator and denotes the order of the derivative. The Riesz-Caputo (RC) is commonly utilized for mathematical modelling of applied problems mainly because it includes initial values that have physical interpretations [3]. For the function F the following holds

$$D^\alpha F = {}^{RC}_{a}D^\alpha_b F = \frac{1}{2}\left({}^{C}_{a}D^\alpha_t F + (-1)^n \, {}^{C}_{t}D^\alpha_b F\right), \tag{5}$$

where a, b, t are so-called terminals, ${}^{C}_{a}D^\alpha_t F$ and ${}^{C}_{t}D^\alpha_b$ denotes left and right sided Caputo derivatives and $n = \lfloor\alpha\rfloor + 1$, where $\lfloor . \rfloor$denotes the floor function.

3. Implementation

Numerical investigations presented in this paper were carried out in Abaqus for three-dimensional cube - single finite element (C3D8R). For the calculations of the strain tensor specialized VUMAT procedure was created and integrated with Abaqus/Explicit solvers. The onset of plastic deformation was described using the HMH yield criterion. As mentioned before, fractional operator introduces new material parameters such as order of the derivative α and terminals a_{ij} and b_{ij} defined in a six-dimensional stress space.

4. Results

In generalized viscoplasticity the relation between the second invariant of strain rate tensor and the second invariant of stress tensor that can be presented as by analogy to [2] as

$$\sqrt{J_2} = \kappa\left[1 + \left(\frac{\sqrt{I_2^P}}{\frac{1}{T_m}\sqrt{\frac{p_{ij}p_{ij}}{2}}}\right)^m\right], \tag{6}$$

where I_2^P is a second invariant of the strain rate tensor, κ is a shear strength, T_m and m are material constants. One can notice that expression in square brackets can acts as a multiplier dependent on the strain-rate that may increase the strength of the material. This approach can be viewed as an elegant and morenatural definition of strain-rate hardening.

In Fig. 1 the evolution of yield strength of the material for $m = 1$ but for changing order of fractional gradient α is presented. It can be noticed that for $\alpha = 1$ classical viscoplasticity is obtained. There is a significant agreement between Fig. 1 and the figure that can found in [2] (Fig. 3). Similar conclusion can be drawn also from analysis Fig. 2 and Fig. 4 from [2] - but in this case non-dimensional parameter is $m = 2$.

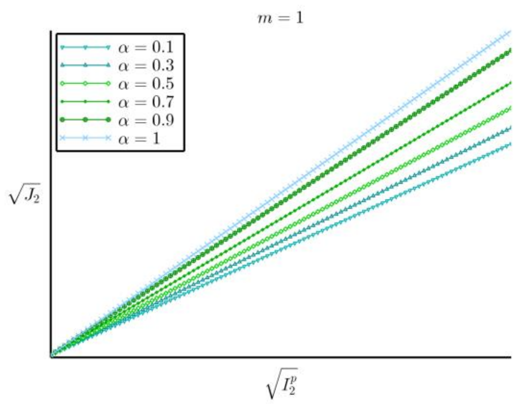

Figure 1: Dependence of $\sqrt{J_2}$ on $\sqrt{I_2^P}$ for $m = 1$

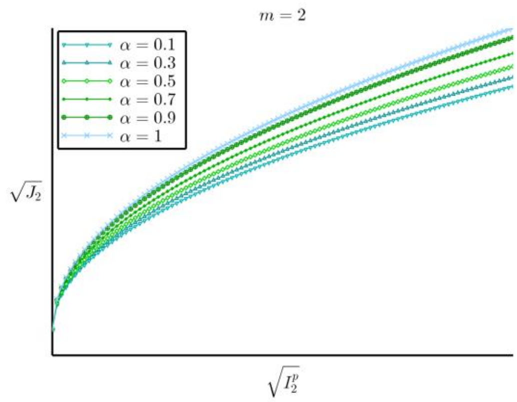

Figure 2: Dependence of $\sqrt{J_2}$ on $\sqrt{I_2^P}$ for $m = 2$

5. Conclusions

Since $m$has the same meaning as parameter δ in Perzyna original paper [2], it is presented that the order of the derivative can be also used to modify the rate effects in the material. Therefore this (fractional) extension, introduces new variable that mayallow a better fit to the data obtained from the experiments.

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