

Dual number algebra method for Green's function derivatives in 3D magneto-electro-elasticity

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Abstract

The Green functions are the basic elements of the boundary element method. To obtain the boundary integral formulation the Green function and its derivative should be known for the considered differential operator. Today the interesting groups of materials are electronic composites. The special case of the electronic composite is the magneto-electro-elastic continuum. The mentioned continuum is a model of the piezoelectric-piezomagnetic composites. The anisotropy of their physical properties makes the problem of Green's function determination very difficult. For that reason Green's functions for the magneto-electro-elastic continuum are not known in the closed form and numerical methods should be applied to determine such Green's functions. These means that the problem of the accurate and simply determination of Green's function derivatives is even harder. Therefore in the present work the dual number algebra method is applied to calculate numerically the derivatives of 3D Green's functions for the magneto-electro-elastic materials. The introduced method is independent on the step size and it can be treated as a special case of the automatic differentiation method. Therefore, the dual number algebra method can be applied as a tool for checking the accuracy of the well-known finite difference schemes. In the final version of the paper the numerical examples and the discussion will be given.

Keywords: Green's function, piezoelectric material, piezomagnetic material, dual number, automatic differentiation

1. Introduction

Green's functions are foundations of several numerical methods, especially the boundary element method [3]. However, the construction of such functions is difficult for modern composite materials [4]. In the paper the magneto-electro-elastic (MEE) composite materials [1, 5, 8] are considered. They exhibit the coupling effects of mechanical and electromagnetic fields, namely the piezoelectric, piezomagnetic and magnetoelectric phenomena. The anisotropy of the MEE materials and the coupling effects do not allow to obtain 3D Green's functions in the closed form. The semi-analytical solution is given by line integral form [1, 5]. The integral can be evaluated by the standard Gaussian integration method, hence the challenge is to calculate the derivatives of the Green functions with high accuracy. The finite difference (FD) scheme could be adopted to calculate the derivatives of Green's functions; after several numerical tests it can be established the appropriate value of the step size, however the value of the optimal step-size is problem-dependent. Therefore, in the present work the dual number algebra method (DNAM) [6, 7] is applied to calculate numerically the derivatives of 3D Green's functions for the MEE materials. The DNAM is practically independent on the step size as shown in [7] and it can be treated as a special case of the automatic differentiation method. These properties make the DNAM an easy-to-implement and highly accurate computational method for the numerical calculation of first-order derivatives of the real functions given by the implicit computational mapping. The presented method is characterized by the extremely high accuracy; hence, it could be also treated as a tool for validating the well-known FD methods, especially for the proper choice of the step size. The mentioned feature is important for the case of the MEE composite models, for which, in general, analytical solutions are not known.

2. Magneto-electro-elastic linear constitutive model

The linear constitutive equations of the magneto-electro-elastic continuum can be expressed as [1, 2, 3]:

$$\begin{aligned}\sigma_{ij} &= c_{ijkl}\varepsilon_{kl} + e_{ijl}(-E_l) + q_{ijl}(-H_l), \\ D_i &= e_{ikl}\varepsilon_{kl} - \kappa_{il}(-E_l) - a_{il}(-H_l), \\ B_i &= q_{ikl}\varepsilon_{kl} - a_{il}(-E_l) - \mu_{il}(-H_l),\end{aligned}\quad (1)$$

where σ_{ij} and ε_{kl} are the elastic stress and strain tensors, respectively; D_i and E_l denote the electric displacement and electric field vectors; B_i and H_l are the magnetic induction and magnetic field vectors; c_{ijkl} , κ_{il} and μ_{il} are the elastic stiffness, the dielectric and magnetic permeability tensors. The elastic field is coupled to the electric and magnetic fields through the piezoelectric e_{ikl} and piezomagnetic q_{ikl} moduli tensors. The electric and magnetic fields are coupled through the magnetoelectric a_{il} moduli tensor. To simplify the notation of equations, generalized quantities can be introduced [3].

In the present formulation, the magneto-electro-elastic composite is modeled as homogeneous, transversal isotropic, linear elastic, linear piezoelectric, linear piezomagnetic and linear magnetoelectric. Other models of the materials can be considered as a special cases, e.g. for the pure piezoelectric material the piezomagnetic and magnetoelectric effect does not occur [8]. The equilibrium equations consist of the mechanical equilibrium equation, the Gauss' law, and the Maxwell equation for the quasi-static magnetic field as shown in [1, 2, 3]. To associate the mechanical strains and the displacement field, the linearized relation of anisotropic elasticity theory is used [1].

3. Integral representation of Green’s function for MEE material

The generalized displacement vector U_{JK} expresses the Green function by the following line integral [1, 5]:

$$U_{JK}(\mathbf{x}) = \frac{1}{8\pi^2 r} \oint_{|\mathbf{n}|=1} \Gamma_{JK}^{-1}(\mathbf{n}) dS(\mathbf{n}), \quad (2)$$

where r is the distance between the source and the field point and Γ_{JK} is the generalized Christoffel tensor [1]. The appropriate parameterization of the unit circle $|\mathbf{n}| = 1$ allows to calculate the integral (2) as a single integral [5]. Because the integral (2) is calculated numerically, the derivative computations is performed by the DNAM, which is also the numerical method.

4. Dual number algebra

The algebra of dual numbers has been originally proposed by Clifford in 1873 [6]. A dual number z is an ordered pair of real numbers (x, y) associated with the real unit 1 and the dual unit ϵ , hence [6, 7]:

$$z = x + y\epsilon. \quad (3)$$

The representation (3) is called Gaussian representation [7]. It is worth to notice, that the dual unit ϵ is an nilpotent number, which means that $\epsilon^2 = 0$ and $\epsilon \neq 0$.

The dual number are elements of the 2-dimensional real algebra generated by real and dual units [6, 7]:

$$D = \mathbb{R}[\epsilon] = \{z = x + y\epsilon \mid (x, y) \in \mathbb{R}^2, (\epsilon^2 = 0) \wedge (\epsilon \neq 0)\}. \quad (4)$$

Basic operation in the dual number algebra are defined by [6]:

$$\begin{aligned} (x_1 + y_1\epsilon) + (x_2 + y_2\epsilon) &= (x_1 + x_2) + (y_1 + y_2)\epsilon, \\ (x_1 + y_1\epsilon) \cdot (x_2 + y_2\epsilon) &= (x_1x_2) + (x_1y_2 + x_2y_1)\epsilon, \\ \forall_{x_2 \neq 0} (x_1 + y_1\epsilon) / (x_2 + y_2\epsilon) &= (x/x_2) + [(x_1y_2 - x_2y_1)/x_2^2]\epsilon. \end{aligned} \quad (5)$$

The special feature of the dual number algebra is an existence of the numbers ϵy , $y \in \mathbb{R}$, which are called the divisors of zero [6, 7]. It is not possible to obtain the inverse element of the number ϵy in the algebra D , given by (4).

A function F of a dual argument $z = x + y\epsilon$ can be represented as [6]:

$$F(z) = f(x, y) + \epsilon g(x, y), \quad (6)$$

where f and g are real functions of real variables x and y . The function $F(z)$ is analytic if [6, 7]:

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}. \quad (7)$$

Therefore, the expansion of the function $F(z)$ in the Taylor series for the dual number argument takes the following form:

$$F(x + y\epsilon) = F(x) + \frac{F'(x)}{1!} y\epsilon. \quad (8)$$

The properties of the dual unit ϵ make that the Taylor series is exactly truncated on the first order term, because for nilpotent numbers $\epsilon^n = 0$, for $n > 1$.

Taking the dual part of the both sides of the equation (8), it is possible to calculate the derivative of the function $F(\cdot)$ as:

$$F'(x) = \frac{\text{Dual}[F(x + y\epsilon)]}{y}, \quad (9)$$

where the operator $\text{Dual}[\cdot]$ extracts the dual part of the dual number. If $y = 1$, then the dual algebra method for calculating the derivative of real functions is equivalent to the automatic differentiation method:

$$F'(x) = \text{Dual}[F(x + 1\epsilon)]. \quad (10)$$

The DNAM could be summarized as following. Defining the dual number operation on the real function $F(\cdot)$, the derivative $F'(\cdot)$ at the point x can be calculated as a dual part of the function $F(\cdot)$ value for the dual argument in the form $x + 1\epsilon$.

The method can be also extended for the functions of several variables. In this case, the ij -th Jacobian’s \mathbf{J} component has the following form:

$$J_{ij} = \text{Dual}[F_i(\mathbf{x} + \epsilon \mathbf{e}_j)], \quad (11)$$

where i -th component of the vector function $\mathbf{F}(\cdot)$ is denoted by $F_i(\cdot)$ and \mathbf{e}_j is the j -th column of the unit matrix \mathbf{I} . $\text{Dim } \mathbf{I} = n$, where n is the number of arguments of the vector function $\mathbf{F}(\cdot)$.

5. Numerical examples

In the full paper the numerical examples will show the robustness and the accuracy of the DNAM for calculations of Green’s functions derivatives in the case of the magneto-electro-elastic continuum. Also, the comparison with the results obtained by the FD schemes will be given to validate the accuracy of the FD methods.

References

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