Effective macroscopic description for heat conduction in periodic composites

Ewaryst Wierzbicki[1] **and Dorota Kula**[2]

[1,2] *Faculty of Civil and Enviromental Engineering, Department of Civil Engineering,*
Warsaw University of Life Sciences,
Nowoursynowska 166, 02-787 Warszawa, Poland
e-mail: el_el@o2.pl, kuladorota@wp.pl

Abstract

The effective description of various macro-phenomena in periodic composites is usually carried out while the diameter of repetitive cells tends to zero. In this contribution we intend to suggest that the asymptotic modeling of effective properties may deal with a limit passage related to an another small parameter. Considerations will be restricted to the simplest case of the thermal effective properties of periodic composites based on the well-known Fourier conduction law. In the proposed approach obtained model equations are a certain alternative form of the well-known heat transfer equation with nonhomogeneous periodic coefficients. Since proposed description depends on the cell size as a parameter, proposed approach yields the scale effect model of heat transfer processes in periodic composites. The limit passage with the cell size to zero leads to the well-known tolerance model as a special case.

Keywords tolerance modelling, orthogonalization approach, Fourier expansion, temperature fluctuations

1. Introduction

The investigation of the temperature field for periodic composites in the form of Fourier expansion or Fourier transform is not possible immediately due to lack of differentiability of the field on the inter-laminae surfaces. But in order to allow spilling Fourier series to perform this role we are to introduce a regular part θ_{reg} of temperature field θ. Residual temperature part θ_{res} in the obtained on this way temperature decomposition

$$\theta = \theta_{res} + \theta_{reg} \tag{1}$$

is supported exclusively on the narrow vicinity $o_\varepsilon(\Gamma)$ of the inter-laminae surfaces Γ,

$$o_\varepsilon(\Gamma) = \{x \in R^3: \quad dist(x,\Gamma) < \varepsilon\}. \tag{2}$$

Hence θ_{res} depends on the parameter $\varepsilon > 0$ but $\theta_{res} = \theta_{res}(\varepsilon)$ should moreover tend to zero while $\varepsilon \to 0$ provided that θ_{reg} is adequately modeled . Under this adequately modelling: 1° θ_{reg} should be represented by the Fourier series

$$\theta_{reg}(y,z,t) = a_0(y,t) + a_p(z,t)\varphi^p(y) \tag{3}$$

valid for an arbitrary particle $x=(y,z)$ from the regular part $\Omega\backslash o_\varepsilon(\Gamma)$ of the region Ω occupying by the periodic composite conductor, 2° for suitable chosen periodic and oscillating Fourier Basis (i.e. $\langle\varphi^p\rangle=0$) and 2° gradient $\nabla\theta_{reg}|$ of θ_{reg} have to vanish in regular points of Γ, i.e. $\nabla\theta_{reg}|_\Gamma=0$. Here and below temperature fields θ_{res}, θ_{reg} as well as θ will be non-dimensional and be related to the reference temperature ϑ_0. Hence nd strictly related to the temperatures $\theta\vartheta_0$, $\theta_{re}\vartheta_{0s}$, $\theta_{reg}\vartheta_0$ are true related temperature fields.

In (1) oscillating orthogonal Fourier basis $\varphi^p = \varphi^p(y)$ should be here strictly connected with the periodicity of the considered conductor, i.e. fields $c = c(y)$ and $K = K(y)$ of the specific heat and heat conductivity tensor should be periodic with respect to the y–variable, referred to as periodic variable. Hence, the region Ω occupied by the composite will be represented in the form $\Omega = \Xi \times \Phi$ for a certain regions $\Phi \subset R^\sigma$, $1 \le \sigma \le 3$, and $\Phi \subset R^{3-\sigma}$. Repetitive cell $\Delta \subset R^\sigma$ and vector basis $\mathbf{p}_1,\ldots,\mathbf{p}_\sigma \in R^3$ determine the basic (repetitive) cell

$$\Delta = \{\textstyle\sum_{i=1}^{\sigma} v_i \mathbf{p}_\sigma : 0 < v_i < 1\} \tag{4}$$

Averaging operation $\langle\cdot\rangle$ used above is taken over the translated repetitive cell $y+\Delta$. Orthogonalization method applied to the well-known heat transfer equation and temperature field representation given by (1) yields $\theta_{res}(\varepsilon) \to 0$ while $\varepsilon \to 0$ and hence averaged temperature field defined as $u = \lim_{\varepsilon\to 0} a_0$ should satisfy the averaged model equation

$$\langle c\rangle\dot{u} - \langle K\rangle\nabla_z{}^T\nabla_z u - \langle K]\nabla_z a_p = -\langle b\rangle \tag{5}$$

in which $\langle K] \equiv \langle K\nabla_\Xi\varphi^p\rangle$ and the evolution of number of Fourier amplitudes a_p are described by equations

$$\begin{aligned}\lambda^2\{\langle c\rangle A_c{}^{pq}\dot{a}_q - A_K{}^{pq}\nabla_z a_q\} + \lambda\langle k\rangle B^{pq}\nabla_z a_q +\\ +\langle k\rangle_H M^{pq}a_q + [K\rangle\nabla_z u = \lambda\langle\varphi^p b\rangle,\end{aligned} \tag{6}$$

$p,q = 1,2,\ldots$. Coefficients $\langle K] \equiv [K\rangle^T$, $M = \langle\nabla_\Xi\bar{\varphi}^p K\nabla_\Xi\bar{\varphi}^q\rangle$, $A_\chi = \langle\bar{\varphi}^p\chi\bar{\varphi}^q\rangle$, $\chi \in \{c,K\}$, are diagonal matrices. For particulars the reader is referred to [1]. *Extended tolerance model of heat conduction in periodic composites* is determined by (4), (5) together with (1), (3).

2. Odd end even fluctuations

The term fluctuation means in this paper every smooth Δ–periodic and oscillating field, $\langle f\rangle = 0$. Moreover,

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$\overline{f} \equiv f + const_f$, $p = 1,2,...$ should be an eigen-function satisfying the following eigen problem:

$$\begin{cases} \partial_z{}^T \partial_z \overline{f}(z) + \mu_s{}^2 \overline{f}(z) = 0, \quad \mu_s{}^2 \geq 0, \quad z \in \Delta_s \geq 0 \\ \nabla_\Xi \overline{f} \big|_\Gamma = 0, \quad \overline{f} \in C^\infty_{per}(\Delta). \end{cases} \tag{7}$$

Even fluctuations are restricted by the additional condition $\underline{f}\,|_\Gamma = 0$, $p = 1,2,...$, while odd fluctuations are those which are not even. Equations (5) for Fourier amplitudes separate onto two disjoint parts for odd $\overline{a}_q$ and even $\underline{a}_q$ Fourier fluctuations. Even Fourier fluctuations are transferred across the composite independently as well on the averaged temperature u as on the odd fluctuations $\overline{a}_q$:

$$\lambda^2\{\langle c\rangle \underline{A}_c{}^{pq} \dot{\underline{a}}_q - \underline{A}_K{}^{pq} \nabla_z \underline{a}_q\} + \lambda\langle k\rangle \underline{B}^{pq} \nabla_z \overline{a}_q + \\ + \underline{M}^{pq} \underline{a}_q = \lambda\langle \underline{\varphi}^p b\rangle, \quad p,q = 1,2,... \quad , \tag{8}$$

Moreover, under $\langle K\nabla_\Xi \underline{\varphi}^p\rangle = 0$, (4) reduces to

$$\langle c\rangle \dot{u} - \langle K\rangle \nabla_z{}^T \nabla_z u - \langle K\nabla_\Xi \overline{\varphi}^p\rangle \nabla_z \overline{a}_p = -\langle b\rangle \tag{9}$$

for odd amplitudes satisfying reduced form of (5)

$$\lambda^2\{\langle c\rangle \overline{A}_c{}^{pq} \dot{\overline{a}}_q - \overline{A}_K{}^{pq} \nabla_z \overline{a}_q\} + \lambda\langle k\rangle \overline{B}^{pq} \nabla_z \overline{a}_q + \\ + \overline{M}^{pq} \overline{a}_q + [K\rangle \nabla_z u = \lambda\langle \overline{\varphi}^p b\rangle, \ p,q = 1,2,..., \tag{10}$$

which under diagonality of A_c, A_K, and M are infinite system of singleton equations, every for other odd amplitude $\overline{a}_q$.

3. Effective conductivity description

Now we shall restrict considerations for stratified conductor and stationary case. In this case we deal with one-directional periodicity for which under (10) every odd amplitudes $\overline{a}$ can be decomposed onto the sum $\overline{a} = \overline{\sigma} + \tilde{a}$ of two parts: a casual fluctuations $\tilde{a} \equiv (\tilde{a}_q)$ and a slave fluctuations $\overline{\sigma} \equiv (\overline{\sigma}_q)$. For a given boundary values $\tilde{a}(z=0) = \tilde{a}^0$ and $\tilde{a}(z=\delta) = \tilde{a}^\delta$ in $\Xi = (0,\delta)$ a casual part of odd fluctuations is given by

$$\overline{a}(z) = \frac{\sinh\sqrt{\overline{A}_K{}^{-1}\overline{M}}\,\dfrac{z}{\lambda}}{\sinh\sqrt{\overline{A}_K{}^{-1}\overline{M}}\,\dfrac{\delta}{\lambda}}\overline{a}^\delta - \frac{\sinh\sqrt{\overline{A}_K{}^{-1}\overline{M}}\,\dfrac{z-\delta}{\lambda}}{\sinh\sqrt{\overline{A}_K{}^{-1}\overline{M}}\,\dfrac{\delta}{\lambda}}\overline{a}^0 \tag{11}$$

while a slave fluctuations are interrelated with the average temperature field u:

$$\overline{\sigma}(z) = \\ = \sqrt{\overline{A}\overline{M}}^{-1} \int_{\xi=0}^{\xi=z} \sinh[\sqrt{\overline{A}^{-1}\overline{M}}(\xi - z)] \frac{du(\xi)}{dz} d\xi \tag{12}$$

Open formula (12) for a slave fluctuations leads to a certain correction of the conductivity macro-modulus, since it modify equation (5) for the averaged temperature field u to the form:

$$\partial \mathcal{K}_\lambda^{eff}[\partial u] = b^{eff} \tag{13}$$

with non-asymptotic effective modulus defined by

$$\mathcal{K}^{eff}{}_\lambda [u] = \mathcal{K}^{eff}{}_0 [u] - \mathcal{K}_\lambda [u] \tag{14}$$

for

$$\mathcal{K}^{eff}{}_0 [u] = \{\langle K\rangle - \langle K]\overline{M}^{-1}[K\rangle\} \frac{du}{dz}$$

$$\mathcal{K}_\lambda [u] = \langle K] \sum\nolimits_{n=1}^{+\infty} (-\lambda \overline{A}\overline{M}^{-1})^n [K\rangle \frac{d^n u}{dz^n} \tag{15}$$

and for

$$b^{eff} = \langle b\rangle + \\ + \langle K]\overline{M}^{-1} \sum\nolimits_{n=1}^{+\infty} (\lambda\sqrt{\overline{A}_K{}^{-1}\overline{M}})^n \{[\cosh\sqrt{\overline{A}_K{}^{-1}\overline{M}}\,\frac{z}{\lambda}][K\rangle \frac{d^n u(0)}{dz^n} + \\ + \lambda\sqrt{\overline{A}_K{}^{-1}\overline{M}}^{-1} [\sinh\sqrt{\overline{A}_K{}^{-1}\overline{M}}\,\frac{z}{\lambda}][K\rangle \frac{d^{n+1}u(0)}{dz^{n+1}}\} \tag{16}$$

It easy to verify that modules $\overline{\mathcal{K}}^{eff}$ and $\overline{\mathcal{K}}_\lambda^{eff}$ are interrelated by

$$\overline{\mathcal{K}}^{eff}[\partial u] = \lim\nolimits_{\lambda\to 0} \overline{\mathcal{K}}_\lambda^{eff}[\partial u] \tag{17}$$

Formula (14) is an extension of the concept the effective conductivity tensor (16) onto the non-asymptotic case.

4. Final remarks

Summarizing considerations it should be emphasized that to make results of this paper ca be treated as complete provided that the solution to the boundary problem for equation (8) similar to that for a casual fluctuations given by (11) will be formulated. To this kind analysis the reader is referred to [3]. Moreover, both extensions of the concept of the effective conductivity tensor, first given by (14) and its asymptotic version (15) have a mathematical structures similar to the structure of the effective modulus tensor proposed in the framework of the tolerance approach given by Cz, Woźniak, cf, [2]. Formula (14) has been obtained by virtue of the concept of the average temperature based on the limit passage with the small parameter ε introduced in the paper. In the previous approaches both concepts of the average temperature and the effective modulus tensor has been strictly connected with the limit passage with the cell diameter to zero, cf. [1].

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