An *hp*-adaptive multiscale FEM for heterogeneous viscoelastic materials

Marek Klimczak[1] **and Witold Cecot**[2*]

[1] *Faculty of Civil Engineering, Cracow University of Technology*
Warszawska 24, 31-155 Cracow, Poland
e-mail: mklimczak@L5.pk.edu.pl

[2] *Faculty of Civil Engineering, Cracow University of Technology*
Warszawska 24, 31-155 Cracow, Poland
e-mail: plcecot@cyf-kr.edu.pl

Abstract

This paper provides a description of the so-called modified multiscale finite element method. Distinguishing two scales - micro- and macroscale we take advantage of the mesh adaptivity. It is used independently at both analysis levels. First, for a coarse mesh generation. Then, at the level of a single coarse mesh element to comply with the microstructure. Using the multiscale FEM we obtain effective stiffness matrices for each of coarse mesh elements. Finally, the main problem is solved using such a coarse discretization with effective matrices incorporating the information on the complex microstructure. Significant reduction of the number of degrees of freedom (NDOF) can be observed with error at the accepted level. Additionally, the method was enhanced with higher order approximation. Proposed approach is particularly beneficial in the context of nonlinear analyses. Application to viscoelastic heterogeneous materials is presented.

Keywords: hp-adaptive FEM, multiscale FEM, Burgers material model

1. Introduction

Among many approaches to numerical modeling of heterogeneous materials one can distinguish a group of discretization-based methods. They are represented for instance by the local numerical homogenization [5, 6] and the multiscale finite element method [1, 4]. In general, the algorithms of such methods are very similar. First, a coarse mesh has to be generated. It is, in fact, the finest mesh we can computationally afford. In our research we take advantage of the *hp*-adaptive FEM code [3] to generate the optimal macroscale mesh. At this level we consider the analyzed domain as a homogeneous one. The second step of the overall algorithm is to proceed independently with each of coarse elements. At this level one accounts for the microstructure heterogeneity. Every coarse element is refined in order to comply with the inclusions distribution. The core of the discretization-based methods is to compute at this level effective stiffness matrices and load vectors. Finally, one solves the coarse mesh problem having transferred the information from the microscale.

2. Problem formulation

Viscoelastic deformations are considered in this paper. Strong formulation of such a problem is as follows:

$$\begin{cases} \mathbf{div}\dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{X}} = \mathbf{0} & \forall t, \ \ \boldsymbol{x} \in \omega_i \subset \Omega \\ \dot{\boldsymbol{\sigma}} = \boldsymbol{C}^{-1}[\dot{\boldsymbol{\varepsilon}}(\dot{\boldsymbol{u}}) - \dot{\boldsymbol{\varepsilon}}^*] & \forall t, \ \ \boldsymbol{x} \in \omega_i \subset \Omega \\ \dot{\boldsymbol{\varepsilon}} = \frac{1}{2}[\nabla\dot{\boldsymbol{u}} + (\nabla\dot{\boldsymbol{u}})^T] & \forall t, \ \ \boldsymbol{x} \in \omega_i \subset \Omega \\ \boldsymbol{\varepsilon}^* = \int_0^t J^*(t-\tau)\dot{\boldsymbol{\sigma}}d\tau & \forall t, \ \ \boldsymbol{x} \in \omega_i \subset \Omega \\ \text{+ initial, boundary} & \\ \text{\& continuity or debonding conditions} & \end{cases} \tag{1}$$

where $\boldsymbol{\sigma}$ denotes the stress tensor, $\boldsymbol{X}$ the body forces, $\boldsymbol{C}^{-1}$ is the tensor of material parameters, J^* is the creep compliance function (kernel). Inelastic strains $\boldsymbol{\varepsilon}^*$ are defined by the Burgers constitutive equation [2]. ω_i is a subdomain with smooth enough tensor $\boldsymbol{C}^{-1}$ and Ω is the analyzed domain.

3. Multiscale finite element method

General idea of the method is as follows. On the basis of the known microstrucure we modify the standard coarse element shape functions. These new functions account for the material distribution and are computed by solving of the following problem defined in every coarse element Ω_i (which we recall after [1]): given $\boldsymbol{\Psi}$, which is a coarse mesh vector-valued shape function, we look for its vector-valued interpolant $\boldsymbol{\Phi}$ that is a discrete solution (a linear combination of polynomial shape functions) of the Dirichlet boundary value problem (2) with $\Phi_k \in C^0(\Omega)$. In other words the new shape functions are linear combinations of fine mesh shape functions. Therefore, whenever the fine mesh approximation provides convergence, also the coarse mesh does.

$$\begin{cases} \frac{\partial}{\partial x_j}\left(C_{ijkl}\frac{\partial \Phi_k}{\partial x_l}\right) = \frac{\partial^2}{\partial x_j x_l}(C^0_{ijkl}\Psi_k) \quad \forall i = 1,2,3, \ \boldsymbol{x} \in \Omega_i \subset \Omega \\ \boldsymbol{\Phi} = \hat{\boldsymbol{\Phi}} \quad on \quad \partial\Omega_i \end{cases} \tag{2}$$

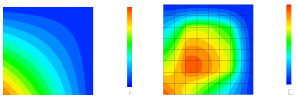
where $\boldsymbol{C^0}$ is the average value of $\boldsymbol{C}$ (a material parameter tensor) in the whole domain Ω.

Knowing fine mesh quantities (denoted with subscript h) one computes $\boldsymbol{K}_H = \boldsymbol{A}^T\boldsymbol{K}_h\boldsymbol{A}$ and $\boldsymbol{f}_H = \boldsymbol{A}^T\boldsymbol{f}_h$ that are effective coarse element stiffness matrix and load vector, respectively. Columns of matrix $\boldsymbol{A}$ are the degrees of freedom of FEM approximation of $\boldsymbol{\Phi}$ - discrete solutions to (2) for each macroscale shape function $\boldsymbol{\Psi}$.

Selected standard and modified coarse element shape func-

* This research was partially financed by Cracow University of Technology under project L5/96/2016/DS

tion is shown in Fig. 1. We analyze a unit cube with a centrally located cube (0.5×0.5×0.5) representing an inclusion. Young moduli of the 'matrix' and the 'inclusion' are equal to $1{\cdot}10^6$ and $1{\cdot}10^7$, respectively. At both scales we use a cubic approximation. On the left side of the Fig. 1 a standard shape function is shown, whereas on the right side an approximated solution to (2) is presented.



$\varphi_{25} \in [0, 0.25]$ $\hat{\varphi}_{25} \in [0, 0.25]$

Figure 1: Selected standard (left column) and modified (right column) coarse element shape function

4. Application of MsFEM to viscoelasticity

The Burgers material model was adopted for modeling of the viscoelastic materials. We assumed also that the material exhibits only small displacements and small displacement gradients. In the algorithm we use in the analysis, occurrence of the inelastic strains contributes to the right hand side vector. Stiffness matrix remains the same.

Coupling nonlinear analyses with the MsFEM is quite easy then. Before executing the main loop over time instants one computes once effective stiffness matrices of the coarse mesh elements. Within the loop we split the viscoelasticity problem into local problems solved within coarse elements (see [6]). On the basis of the inelastic strains one computes fine mesh load vector that is subsequently multiplied by the restriction operator $\boldsymbol{R}$ ($\boldsymbol{R} = \boldsymbol{A}^T$) to constitute an effective load vector. Finally, one can solve the coarse mesh viscoelasticity problem using effective element matrices.

5. hp-adaptivity

Additional enhancement of the proposed approach is the application of the automatic adaptivity. We take advantage of the code described in [3]. Considering the whole domain we solve an auxiliary problem in order to generate the optimal hp-adapted coarse mesh (see [3]) accounting for the global solution behavior but neglecting inelastic phenomena. We keep this mesh for the viscoelasic problem without further refinements. Additionally, within coarse mesh elements h-adaptivity detecting phase boundaries is performed. These mesh refinements are not repeated within viscoelastic loops. Such an approach allows one to distribute computational power effectively.

Results of an exemplary creep test are shown in Fig. 2. We analyzed a brick (2m×1m×1m) fixed at the right hand side face and loaded at the opposite one. There are 128 periodically distributed cuboid inclusions with Young modulus values twice as large as the matrix. In Fig. 3 one can observe an hp-adapted coarse mesh (third and fourth order of approximation). The comparison of the dominating strain component at selected point A for the MsFEM and the brute force (over-kill mesh) solutions is shown in Fig. 2.

6. Final remarks

Novelty of the proposed approach is the integration of two well-established efficient methods (MsFEM and hp-FEM) for modeling of heterogeneous viscoelastic materials. Numerical results (see [7]) confirmed high reliability of our approach. Particularly, application of the higher order approximation due to automatic hp-adaptivity increased the efficiency. Enormous reduction of NDOF could be observed comparing homogenized solutions and fine mesh brute force solutions. Error of the MsFEM was acceptable - of the order of several percent.

Presented approach is quite general. There are no limitations concerning periodicity of the domain, nor separation of scales condition that is typical in RVE-based homogenization.

Our further research effort is to enhance the implementation towards full parallelization of the overall method.

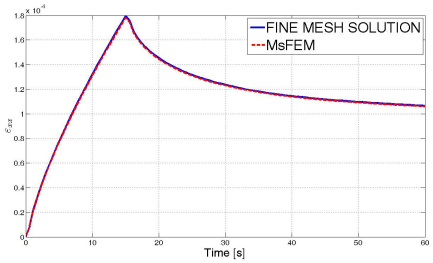


Figure 2: ε_{xx} at point A

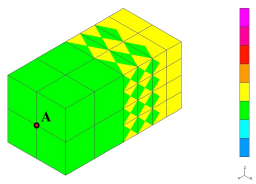


Figure 3: hp-adapted coarse mesh

References

[1] Cecot, W. and Oleksy, M., High order FEM for multigrid homogenization, *Computer Methods in Applied Mechanics and Engineering*, Vol. 70, No. 7, pp. 1391–1400, 2015.

[2] Collop, A.C., Scarpas, A.T., Kasbergen, C. and de Bondt, A., Development and finite element implementation of a stress dependent elasto-visco-plastic constitutive model with damage for asphalt, *Transportation Resarch Record*, Vol. 1832, pp. 96–104, 2003.

[3] Demkowicz, L., Kurtz, J., Pardo, D., Paszyński, M., Rachowicz, W. and Zdunek, A., *Computing with hp-Adaptive Finite Element Method, Volume 2, Frontiers: Three Dimensional Elliptic and Maxwell Problems with Applications*, Chapmann and Hall, 2008.

[4] Efendiev, Y. and Hou, T., *Multiscale Finite Element Methods. Theory and Applications*, Springer, 2009.

[5] Jhurani, C.K. and Demkowicz, L., Multiscale modeling using goal-oriented adaptivity and numerical homogenization. Part I: Mathematical formulation and numerical results, *Computer Methods in Applied Mechanics and Engineering*, Vol. 213–216, pp. 399–417, 2012.

[6] Klimczak, M. and Cecot, W., Application of Local Numerical Homogenization and hp-Adaptive FEM for Modeling of Heterogeneous Viscoelastic Materials, *Engineering Transactions*, Vol. 63, No. 3, pp. 317–327, 2015.

[7] Klimczak, M. and Cecot, W., Comparison of Two Methods for Numerical Upscaling, *Engineering Transactions*, Vol. 63, No. 3, pp. 317–327, 2016.