Customizing the engineering moduli of elasticity in the context of structural optimization

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Abstract

The research concerns a structural optimization problem set in the broad perspective of Free Material Design (FMD). Our approach differs from the standard one by choice of the design variable. Namely, we drop the full Hooke's tensor, focusing on engineering moduli of elasticity instead. Based on discretion in the representation of stress, the stored elastic energy is appropriately decomposed, thus elastic moduli corresponding to that decomposition are explicitly revealed. This paves the way towards customizing the moduli in the context of structural optimization. Hence, we propose a term Engineering Moduli Design (EMD) for the nickname of the method. The general idea of EMD is presented in the framework of three-dimensional linear elasticity for a body subjected to one-parameter load. The method, however, is immediately applicable in two dimensions, as shown in the example. It can also be adapted to optimal design of plates in bending, coupled membrane-bending optimization problems, and multi-parameter load cases.

Keywords: optimal design, Engineering Moduli Design, optimal Young's modulus, optimal shear modulus, optimal bulk modulus

1. Notation and definitions

Consider a body occupying a domain $\Omega \subset \mathbb{R}^3$ and parameterize Ω by $x = (x_1, x_2, x_3)$. Let $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ represent the displacement field in Ω and write $\boldsymbol{u} \in V$ if the components of $\boldsymbol{u}$ satisfy certain conditions on the boundary $\partial\Omega$. The deformation measure $\boldsymbol{\varepsilon}(\boldsymbol{u}) : \Omega \to \mathbb{E}^2_s$ is given by $\varepsilon_{\alpha\beta}(\boldsymbol{u}) = (1/2)(u_{\alpha,\beta} + u_{\beta,\alpha})$, $\alpha, \beta = 1, 2, 3$. Here, $\mathbb{E}^2_s$ denotes the space of second-order symmetric tensors.

The elastic material distributed in Ω is anisotropic and nonhomogeneous. Write $\mathbb{E}^4_s$ for the set of fourth-order tensors with Hooke's symmetries and assume $\{\mathbf{C},\, \mathbf{C}^{-1}\} : \Omega \to \mathbb{E}^4_s$ for the stiffness and compliance tensor fields respectively.

Introduce the stress field $\boldsymbol{\tau} : \Omega \to \mathbb{E}^2_s$ and suppose that $\boldsymbol{\tau}$ is linked with the deformation measure by $\boldsymbol{\varepsilon} = \mathbf{C}^{-1}\boldsymbol{\tau}$. Next, suppose that the body is subjected to a one-parameter load applied on the boundary $\partial\Omega$, i.e. introduce $\boldsymbol{p} : \partial\Omega \to \mathbb{R}^3$, and express the equilibrium equation as

$$\int_\Omega \boldsymbol{\tau} : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \int_{\partial\Omega} \boldsymbol{p} \cdot \boldsymbol{v}\, ds, \quad \forall \boldsymbol{v} \in V. \tag{1}$$

The symbols ":" and "·" denote scalar products in respective spaces. We write $\boldsymbol{\tau} \in \Sigma$ for the stress fields satisfying (1).

In case of linear elasticity, the (doubled) stored energy function $\Phi = \Phi(\mathbf{C}, \boldsymbol{\tau})$ is given by the quadratic form

$$\Phi(\mathbf{C}, \boldsymbol{\tau}) = \boldsymbol{\tau} : \mathbf{C}^{-1}\boldsymbol{\tau}. \tag{2}$$

Set $\boldsymbol{m} \in \mathbb{R}^3$, $\|\boldsymbol{m}\| = 1$, for a direction in physical space $\mathbb{R}^3$; $(\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3)$ – for an orthonormal triplet, and $\mathbf{1}$ – for a unit tensor in $\mathbb{E}^4_s$. We discuss the optimality of: $E(\boldsymbol{m})$ – Young's modulus in direction $\boldsymbol{m}$; $\nu(\boldsymbol{m}_\alpha, \boldsymbol{m}_\beta)$ – the Poisson ratio in direction $\boldsymbol{m}_\alpha$ corresponding to the stretch in direction $\boldsymbol{m}_\beta$; $G(\boldsymbol{m}_\alpha, \boldsymbol{m}_\beta)$ – the shear modulus in the plane defined by a pair $(\boldsymbol{m}_\alpha, \boldsymbol{m}_\beta)$ and K – the bulk modulus. By [5] we have

$$\frac{1}{E(\boldsymbol{m})} = (\boldsymbol{m} \otimes \boldsymbol{m}) : \mathbf{C}^{-1}(\boldsymbol{m} \otimes \boldsymbol{m}), \tag{3}$$

$$\frac{\nu(\boldsymbol{m}_\alpha, \boldsymbol{m}_\beta)}{E(\boldsymbol{m}_\beta)} = -(\boldsymbol{m}_\alpha \otimes \boldsymbol{m}_\alpha) : \mathbf{C}^{-1}(\boldsymbol{m}_\beta \otimes \boldsymbol{m}_\beta), \tag{4}$$

$$\frac{1}{4G(\boldsymbol{m}_\alpha, \boldsymbol{m}_\beta)} = (\boldsymbol{m}_\alpha \otimes \boldsymbol{m}_\beta) : \mathbf{C}^{-1}(\boldsymbol{m}_\beta \otimes \boldsymbol{m}_\alpha), \tag{5}$$

$$\frac{1}{K} = \mathbf{1} : \mathbf{C}^{-1}\mathbf{1}. \tag{6}$$

In what follows, we write $E_\alpha \equiv E(\boldsymbol{m}_\alpha)$, $\nu_{\alpha\beta} \equiv \nu(\boldsymbol{m}_\alpha, \boldsymbol{m}_\beta)$ and $G_{\alpha\beta} \equiv G(\boldsymbol{m}_\alpha, \boldsymbol{m}_\beta)$. For positively defined $\Phi(\mathbf{C}, \boldsymbol{\tau})$, and due to Hooke's symmetries of $\mathbf{C}$ we have

$$E_\alpha \geq 0, \quad E_\alpha\, \nu_{\alpha\beta} = E_\beta\, \nu_{\beta\alpha}, \quad G_{\alpha\beta} \geq 0, \quad K \geq 0. \tag{7}$$

2. Customizing Young's modulus

Fix $\{\boldsymbol{m}_\alpha\}^3_{\alpha=1}$ for a proper basis of $\boldsymbol{\tau}$ at $x \in \Omega$, and write

$$\boldsymbol{\tau} = \sum_{\alpha=1,2,3} \tau_\alpha\, \boldsymbol{m}_\alpha \otimes \boldsymbol{m}_\alpha. \tag{8}$$

With (3) and (4), the stored energy decomposes to

$$\begin{aligned}\Phi(\mathbf{C}, \boldsymbol{\tau}) &= \Phi_E(\mathbf{C}, \boldsymbol{\tau}) + \Phi_{E\nu}(\mathbf{C}, \boldsymbol{\tau}) \\ &= \sum_{\alpha=1,2,3} \frac{(\tau_\alpha)^2}{E_\alpha} - \sum_{\substack{\alpha,\beta=1,2,3 \\ \alpha<\beta}} (\tau_\alpha \tau_\beta)\Big(\frac{\nu_{\alpha\beta}}{E_\beta} + \frac{\nu_{\beta\alpha}}{E_\alpha}\Big).\end{aligned} \tag{9}$$

Optimal $E_\alpha = E_\alpha(x)$, $\alpha = 1, 2, 3$, are determined through

$$(P_E)\ \left|\ \begin{aligned} &Y = \min\left\{ \int_\Omega \Phi(\mathbf{C}, \boldsymbol{\tau})\, dx \ \middle|\ \boldsymbol{\tau} \in \Sigma,\ E_\alpha \in \mathcal{E} \right\} \\ &\mathcal{E} = \left\{ E_\alpha \,:\, E_\alpha \geq 0, \sum_{\alpha=1,2,3} \int_\Omega E_\alpha\, dx = E_0\, |\Omega| \right\}. \end{aligned}\right.$$

A close link between (P_E) and celebrated Michell problem of minimal structural weight unveils in case of $\nu_{\alpha\beta} = 0$, $\alpha, \beta = 1, 2, 3$, $\alpha \neq \beta$. This feature is discussed in [3].

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3. Customizing shear and bulk moduli

Assume now that stress is represented by,

$$\boldsymbol{\tau} = \tau\,\mathbf{1} + \boldsymbol{d}, \quad \tau = \frac{\operatorname{tr}\boldsymbol{\tau}}{3}, \quad \boldsymbol{d} = \boldsymbol{s}_1 + \boldsymbol{s}_2 + \boldsymbol{s}_3 \tag{10}$$

where $\boldsymbol{d}$ denotes a deviator of $\boldsymbol{\tau}$ and

$$\begin{aligned}
\boldsymbol{s}_1 &= s_1\,(\boldsymbol{m}_2 \otimes \boldsymbol{m}_3 + \boldsymbol{m}_3 \otimes \boldsymbol{m}_2),\\
\boldsymbol{s}_2 &= s_2\,(\boldsymbol{m}_1 \otimes \boldsymbol{m}_3 + \boldsymbol{m}_3 \otimes \boldsymbol{m}_1),\\
\boldsymbol{s}_3 &= s_3\,(\boldsymbol{m}_1 \otimes \boldsymbol{m}_2 + \boldsymbol{m}_2 \otimes \boldsymbol{m}_1),
\end{aligned} \tag{11}$$

are pure shear stress tensors acting in the planes defined by pairs $(\boldsymbol{m}_\alpha, \boldsymbol{m}_\beta)$, $\alpha, \beta = 1, 2, 3$, $\alpha \neq \beta$. For given $\boldsymbol{\tau}$, the orientation of $(\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3)$ in $\mathbb{R}^3$ can be exactly found at each $x \in \Omega$. The Reader is referred to [1, 4] for details.

The stored energy is now decomposed in the form

$$\Phi(\mathbf{C}, \boldsymbol{\tau}) = \Phi_G(\mathbf{C}, \boldsymbol{\tau}) + \Phi_K(\mathbf{C}, \boldsymbol{\tau}) + \Phi_0(\mathbf{C}, \boldsymbol{\tau}) \tag{12}$$

where, see (5) and (6),

$$\Phi_G(\mathbf{C}, \boldsymbol{\tau}) = \sum_{\alpha=1,2,3} \boldsymbol{s}_\alpha : \mathbf{C}^{-1}\boldsymbol{s}_\alpha = \frac{(s_1)^2}{4G_{23}} + \frac{(s_2)^2}{4G_{13}} + \frac{(s_3)^2}{4G_{12}}, \tag{13}$$

$$\Phi_K(\mathbf{C}, \boldsymbol{\tau}) = \tau^2\,\mathbf{1} : \mathbf{C}^{-1}\mathbf{1} = \frac{\tau^2}{K}, \tag{14}$$

$$\begin{aligned}
\Phi_0(\mathbf{C}, \boldsymbol{\tau}) &= \tau\,\big(\boldsymbol{d} : \mathbf{C}^{-1}\mathbf{1} + \mathbf{1} : \mathbf{C}^{-1}\boldsymbol{d}\big)\\
&\quad + \sum_{\substack{\alpha,\beta=1,2,3\\ \alpha<\beta}} \big(\boldsymbol{s}_\alpha : \mathbf{C}^{-1}\boldsymbol{s}_\beta + \boldsymbol{s}_\beta : \mathbf{C}^{-1}\boldsymbol{s}_\alpha\big).
\end{aligned} \tag{15}$$

Note that $\Phi_0(\mathbf{C}, \boldsymbol{\tau}) = 0$ for orthotropic, tetragonal, cubic, transversely isotropic and isotropic materials. Indeed, the coefficients of $\mathbf{C}$ involved in (15) take zero values for these symmetry classes, see [2].

Optimal $G_{\alpha\beta} = G_{\alpha\beta}(x)$, $\alpha, \beta = 1, 2, 3$, are given by

$$(P_G)\quad \left|\begin{array}{l}
Y = \min\left\{ \displaystyle\int_\Omega \Phi(\mathbf{C}, \boldsymbol{\tau})\,dx \;\middle|\; \boldsymbol{\tau} \in \Sigma,\ G_{\alpha\beta} \in \mathcal{G} \right\}\\
\mathcal{G} = \left\{ G_{\alpha\beta} : G_{\alpha\beta} \geq 0, \displaystyle\sum_{\substack{\alpha,\beta=1,2,3\\ \alpha<\beta}} \int_\Omega G_{\alpha\beta}\,dx = G_0|\Omega| \right\}.
\end{array}\right.$$

Similarly, optimal $K = K(x)$ is obtained from

$$(P_K)\quad \left|\begin{array}{l}
Y = \min\left\{ \displaystyle\int_\Omega \Phi(\mathbf{C}, \boldsymbol{\tau})\,dx \;\middle|\; \boldsymbol{\tau} \in \Sigma,\ K \in \mathcal{K} \right\}\\
\mathcal{K} = \left\{ K : K \geq 0,\ \displaystyle\int_\Omega K\,dx = K_0\,|\Omega| \right\}.
\end{array}\right.$$

4. Example

The discussion from previous sections can be repeated, with elementary changes, in the two-dimensional setting. Purpose of an example in this section is illustrative: to provide a brief insight into the problems formulated in the framework of EMD.

Consider annular plate of internal radius $r_{\text{int}} = a$, external radius $r_{\text{ext}} = b$, $a < b$, and constant thickness $h \ll a$. Assume that uniform, tangent loading τ_0 is applied to the internal face of the plate; the external face is loaded with tangent loading of such intensity that the plate is in statical equilibrium. Introduce polar coordinates (ϑ, ϱ), where

$$\varrho = \frac{r}{a} \in (1, \gamma), \quad \gamma = \frac{b}{a} > 1, \tag{16}$$

and $\{\boldsymbol{e}_\vartheta, \boldsymbol{e}_\varrho\}$ – a local, orthonormal basis.

Thus posed problem is statically determinate with the stress field $\boldsymbol{\tau} = \boldsymbol{\tau}(\varrho)$,

$$\boldsymbol{\tau}(\varrho) = \frac{\tau_0}{\varrho^2}(\boldsymbol{e}_\vartheta \otimes \boldsymbol{e}_\varrho + \boldsymbol{e}_\varrho \otimes \boldsymbol{e}_\vartheta). \tag{17}$$

Optimal distribution of Young's modulus. Write $(\boldsymbol{m}_1, \boldsymbol{m}_2)$ for the curvilinear proper basis of $\boldsymbol{\tau}$. The problem (P_E) becomes

$$Y = \int_0^{2\pi}\!\!\int_1^{\gamma} \min_{E_1,E_2\geq 0} \Phi_\lambda(\mathbf{C}, \boldsymbol{\tau})\,a^2\varrho\,d\varrho\,d\vartheta - \lambda E_0\,\pi\,a^2(\gamma^2 - 1),$$

$$\Phi_\lambda(\mathbf{C}, \boldsymbol{\tau}) = \frac{\tau_0}{\varrho^2}\Big(\frac{1+\nu_{21}}{E_1} + \frac{1+\nu_{12}}{E_2}\Big) + \lambda(E_1 + E_2), \tag{18}$$

where λ denotes the Langrange multiplier for the design constraint (limit on the cost function) imposed in the description of $\mathcal{E}$. From the necessary conditions of optimality we obtain

$$E_1 = \frac{|\tau_0|}{\varrho^2}\sqrt{\frac{1+\nu_{21}}{\lambda}}, \quad E_2 = \frac{|\tau_0|}{\varrho^2}\sqrt{\frac{1+\nu_{12}}{\lambda}}. \tag{19}$$

Restriction $(7)_2$ gives $\nu_{12}\sqrt{1+\nu_{21}} = \nu_{21}\sqrt{1+\nu_{12}}$ and it is immediate that $\nu_{12} = \nu_{21} = \nu,\ \nu \geq -1$ in the problem at hand.

Calculating λ from the cost function leads to optimal distribution $E_1^* = E_2^* = E^*$, $E^* = E^*(\varrho)$,

$$E^*(\varrho) = \frac{E_0\,(\gamma^2 - 1)}{4\ln\gamma}\,\frac{1}{\varrho^2}. \tag{20}$$

Note that the "zero cost" structure is theoretically possible. Indeed, setting $E_0 = 0$ in the isoperimetric condition and making use of (19) and (20) leads to the nontrivial, yet peculiar, characteristics of the optimal material, namely $\nu = -1$ and $E_1 = E_2 = 0$.

Optimal distribution of shear modulus. Set now $\boldsymbol{m}_1 \equiv \boldsymbol{e}_\vartheta$ and $\boldsymbol{m}_2 \equiv \boldsymbol{e}_\varrho$ in (17) and note that $\boldsymbol{\tau}$ is a pure shear stress tensor. Hence, $\Phi_K = \Phi_0 = 0$ in (14) and (15) thus $\Phi = \Phi_G(\mathbf{C}, \boldsymbol{\tau})$.

With $G_{12} \equiv G$, the problem (P_G) becomes

$$Y = \int_0^{2\pi}\!\!\int_1^{\gamma} \min_{G\geq 0} \Phi_\lambda(\mathbf{C}, \boldsymbol{\tau})\,a^2\varrho\,d\varrho\,d\vartheta - \lambda G_0\,\pi\,a^2(\gamma^2 - 1),$$

$$\Phi_\lambda(\mathbf{C}, \boldsymbol{\tau}) = \frac{\tau_0}{4G\,\varrho^2} + \lambda G. \tag{21}$$

The necessary condition of optimality and the integral constraint in the description of $\mathcal{G}$ give $G^* = G^*(\varrho)$,

$$G^*(\varrho) = \frac{G_0\,(\gamma^2 - 1)}{2\ln\gamma}\,\frac{1}{\varrho^2}. \tag{22}$$

5. Remark

To compare the FMD and EMD approaches, let us note that FMD optimizes $\mathbf{C}$ in all directions $\boldsymbol{\omega} \in \mathbb{E}_s^2$, while EMD limits the search to $\boldsymbol{\omega} = \boldsymbol{m} \otimes \boldsymbol{m}$ (Young's modulus); $\boldsymbol{\omega} = \boldsymbol{m}_\alpha \otimes \boldsymbol{m}_\beta + \boldsymbol{m}_\beta \otimes \boldsymbol{m}_\alpha$ (shear modulus) and $\boldsymbol{\omega} = \mathbf{1}$ (bulk modulus).

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