# A compact algorithm for semi-analytical computing of internal forces in RC cross-section

**Aleksander Matuszak**[1] **and Piotr Pluciński**[2]

[1] *Institute for Computational Civil Engineering, Cracow University of Technology*
*Warszawska 24, 31-155 Krakow, Poland*
*e-mail: a.matuszak@l5.pk.edu.pl*

[2] *Institute for Computational Civil Engineering, Cracow University of Technology*
*Warszawska 24, 31-155 Krakow, Poland*
*e-mail: p.plucinski@l5.pk.edu.pl*

Abstract

Integration of stress in RC cross-section is an important step in any computation of an RC beam or frame. The standard procedure in most cases is to use numerical integration. However, in 2005, Zupan & Saje developed a semi-analytical method of section stress integration [4].

This paper discusses a compact algorithm for the method published by Zupan & Saje. Compactness is achieved by splitting computations into two parts. One is computing stress integrals that are independent of the section geometry and the other is computing coefficients dependent on geometry but not on stress.

In order to demonstrate the steps of the algorithm, the authors have provieded a detailed example of the most frequently studied section, published by Fafitis [1]. The second part presents the complete algorithm.

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## 1. Introduction

Internal forces – e.g. normal force and bending moments – of a bar cross-section are the result of the stress field integrated over the section area.

The subroutine for stress integration for a given state of deformation is the one of most important subroutines in computing bar structures, such as beams or frames; particularly, for RC as a highly non-linear material.

This subroutine makes it possible to solve the reverse, non-linear problem iteratively: for given external forces to find the deformation that provides equilibrium. The subroutine for equilibrium enables development of other subroutines, e.g. one for computing computes the envelope: i.e. to find the line (capacity contour) which can be considered as a set of ultimate limit state (ultimate strength) points.

Analytical integration formulae may be used for simple section shapes, such as rectangular, T-shape or similar. For complex section geometries – sections may have holes or can consist of separate parts – the boundary integral approach is necessary, which involves evaluation of a series of one variable integrals. In most cases, the integrals are computed with the use of numerical integration (e.g. [3], [2]).

Another possibility is to use the method developed by Zupan & Saje [4] which does not require numerical integration. We consider it as semi-analytical since it requires a substantial amount of computations. It is rather difficult to use in manual calculations.

Moreover, this method – although conceptually simple – seems to be quite complex in practical use. Perhaps this is the reason why it has failed to replace numerical integration.

The paper demonstrates sequence of computation on the example published by Fafitis [1]. The complete algorithm is formulated in the last part.

## 2. Internal forces

The method of Zupan & Saje imposes two limitations: the strain may not be uniform and the boundary of the cross-section has to be composed of (or approximated by) straight segments.

### 2.1. System of coordinates

Computations have to be done in a system of coordinates $(r, t)$ rotated in such a way that the field of strain is independent of $r$.

### 2.2. Steel

Internal forces of RC section are composed of a steel part and a concrete part. The steel part may be computed as follows:

$$S_i = \sum_{\text{bars}} A r^{\alpha} t^{\beta} \sigma(\varepsilon) \tag{1}$$

where $S_i$ is the section force component, $A$ is the area of given reinforcement bar and $\alpha$, $\beta$ depends of component of internal force. In what follows the steel part $S_i$ will not be considered.

### 2.3. Concrete

The concrete part of internal forces is represented by boundary intergals. The integrals may be written in the form:

$$F_i = \oint w_i(t)\sigma(t)dt \tag{2}$$

where $F_i$ is the section force component $[N, M_r, M_t]$ and $w_i(t)$ is the polynomial of variable $t$, appropriate for the given component.

The closed integral (2) may be split into a sum of integrals over pieces of boundary. As long as the section boundary is composed of straight edges the strain is linear along each edge i.e. function $\varepsilon(t)$ is linear. Therefore – as pointed out by Zupan & Saje [4] – the inverse function $t(\varepsilon)$ is also linear and $\varepsilon$ may be

Table 1: Numerical results of stress integration according to the Zupan & Saje approach. Normal force in kips.

| | | | | |
|---|---|---|---|---|
| | | Edge 1-2 | | |
| $D \cdot [A_1\ B_1\ C_1]$ | -0.000000e+00 | -6.155597e+07 | -1.891938e+05 | |
| $I_\alpha$ | 5.924722e-08 | -2.400624e-05 | 1.087326e-02 | $\Delta N =$-579.426 |
| | | Edge 2-3 | | |
| $D \cdot [A_1\ B_1\ C_1]$ | -0.000000e+00 | 6.155597e+07 | 2.416980e+05 | |
| $I_\alpha$ | -5.377954e-08 | 1.989080e-05 | -7.650901e-03 | $\Delta N =$-624.81 |
| | | Edge 3-4 | | |
| $D \cdot [A_1\ B_1\ C_1]$ | -0.000000e+00 | -6.155597e+07 | 3.271789e+04 | |
| $I_\alpha$ | -5.826715e-09 | 4.723782e-06 | -4.390765e-03 | $\Delta N =$-434.434 |
| | | Edge 4-1 | | |
| $D \cdot [A_1\ B_1\ C_1]$ | -0.000000e+00 | 6.155597e+07 | -9.116942e+04 | |
| $I_\alpha$ | -3.590325e-10 | 6.083446e-07 | -1.168407e-03 | $\Delta N =$-143.97 |

used as a variable in integral (2).

The contribution from $n$-th edge may be written as follows:

$$\Delta F_i^{(n)} = + \int_{\varepsilon_{k-1}}^{\varepsilon_k} W_i(\varepsilon)\sigma(\varepsilon)\frac{dt}{d\varepsilon}d\varepsilon \tag{3}$$

where $\frac{dt}{d\varepsilon} = D$ is constant.

The polynomials $W_i(\varepsilon)$ are at most quadratic and they may be written as follows:

$$W_i(\varepsilon) = A_i\varepsilon^2 + B_i\varepsilon + C_i \tag{4}$$

Substituting this into (3) leads to the expression with three integrals of the following type:

$$I_\alpha = \int_{\varepsilon_{k-1}}^{\varepsilon_k} \varepsilon^\alpha \sigma(\varepsilon)d\varepsilon \tag{5}$$

where $\alpha = 0, 1, 2$.

The dependence on geometry of the edge is now represented by coefficients $A_i, B_i, C_i$. Integrals of type (5) may be expressed as $I_\alpha = H_\alpha(\varepsilon_k) - H_\alpha(\varepsilon_{k-1})$ where:

$$H_\alpha(\varepsilon_p) = \int_0^{\varepsilon_p} \varepsilon^\alpha \sigma(\varepsilon)d\varepsilon \tag{6}$$

and this integral is dependent only on the stress-strain law and the upper limit $\varepsilon_p$ but it is not dependent on the edge geometry.

### *2.4. Sequence of computations*

For each vertex strain $\varepsilon_i$ and the set of $H_\alpha(\varepsilon_i)$ must be computed. Then, the set of $A_i, B_i, C_i, D$ as well as $I_\alpha$ has to be computed for each edge. The value of integral (2) may be obtained by combining the apropriate $I_\alpha$ with $A_i, B_i, C_i, D$.

## 3. Example

The example published by Fafitis [1] shown in Fig. 1 is a kind of reference example for internal forces computation. Therefore, it has been deemed convenient to demonstrate the discussed approach on this example.

The stress-strain law is a parabola-rectangle, shown on the right in Fig. 1. The obtained results are shown in Table 1. For the sake of brevity only the computation of $N$ is shown. Row $I_\alpha$ are ordered from the highest to lowest, i.e. the sequence of $I_\alpha$ is $I_2$, $I_1$ and $I_0$.

The dot product of row $D \cdot [A_1\ \ B_1\ \ C_1]$ and row $I_\alpha$ gives $\Delta N$ as shown in the last column. The sum of $\Delta N$ for all edges gives the normal force i.e. $N$=-1782.64 kip for the section.

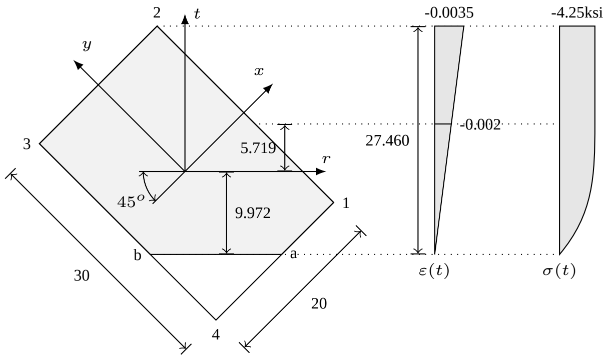


Figure 1: The rectangular section (dimensions in inches) analysed by Fafitis and distribution of strain and stress in concrete

## 4. Algorithm

The sequence of computations shown above is not the algorithm yet, since it is necessary to take into account special cases as well. For example, it is not possible to write function $r(t)$ for the horizontal edge of the section.

Moreover, in the approach of Zupan & Saje strain cannot be uniform in the section area. Designed algorithm overcomes this limitation.

## 5. Conclusions

Zupan & Saje method simplifies integration of section forces for the wide range of nonlinear stress-strain laws. Designed algorithm makes implementation of this approach simple and robust.

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