# Metamorphoses of amplitude curves in a system of coupled oscillators: the case of degenerate singular points 

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#### Abstract

We study dynamics of two coupled periodically driven oscillators. The internal motion is separated off exactly to yield a nonlinear fourth-order equation describing inner dynamics. Periodic steady-state solutions of the fourth-order equation are determined within the Krylov-Bogoliubov-Mitropolsky (KBM) approach and the corresponding amplitude profiles are computed.

Metamorphoses of the corresponding amplitude curves induced by changes of control parameters and the corresponding changes of dynamics are studied. In the present paper we investigate changes of dynamics near degenerate singular points of the amplitude profiles.


Keywords: coupled oscillators, amplitude profiles, degenerate singular points

## 1. Introduction

We study dynamics of two coupled nonlinear oscillators, one of which is driven by an external periodic force. Equations of motion are:

$$
\left.\begin{array}{l}
m \ddot{x}-V(\dot{x})-R(x)+V_{e}(\dot{y})+R_{e}(y)=F(t)  \tag{1}\\
m_{e}(\ddot{x}+\ddot{y})-V_{e}(\dot{y})-R_{e}(y)=0
\end{array}\right\}
$$

where $x \equiv x_{1}$ is position of primary mass $m, y \equiv x_{2}-x_{1}$ is relative position of another mass $m_{e}$ attached to $m$ at position $x_{2}$ and $R, V$ and $R_{e}, V_{e}$ are nonlinear elastic restoring force and nonlinear force of internal friction for masses $m, m_{e}$, respectively (we use convention $\dot{x} \equiv \frac{d x}{d t}$, etc.). Dynamics of coupled, periodically driven oscillators, is very complicated. Dynamic vibration absorber is a typical mechanical model described by (1) (in this case $m$ is usually much larger than $m_{e}$ ) [1].

In the present work we continue study of the nonlinear case studied in Ref. [3] which can be reduced to exact 4 -th order differential equation for internal motion (i.e. motion of the small mass $m_{e}$ ). In [3] we have computed for the first time a resonance curve with a degenerate singular point. We investigate now the nature of some degenerate singular points and the corresponding metamorphoses of the amplitude profiles and dynamics.

## 2. Equations of motion

In what follows we assume as in Ref. [3]:

$$
\left.\begin{array}{ll}
R(x)=-\alpha x, & R_{e}(y)=-\alpha_{e} y-\gamma_{e} y^{3}  \tag{2}\\
V(\dot{x})=-\nu \dot{x}, & V_{e}(\dot{y})=-\nu_{e} \dot{y}-\beta_{e} \dot{y}^{3}
\end{array}\right\}
$$

and $F(t)=f \cos (\omega t)$. It is possible to separate off variable $y$ to obtain the following exact equation for relative motion [3]:
$\widehat{L}\left(\mu \ddot{y}-V_{e}(\dot{y})-R_{e}(y)\right)+\epsilon m_{e} \widehat{K} y=F \cos (\omega t)$
where $\widehat{L}=M \frac{d^{2}}{d t^{2}}+\nu \frac{d}{d t}+\alpha, \widehat{K}=\left(\nu \frac{d}{d t}+\alpha\right) \frac{d^{2}}{d t^{2}}, F=m_{e} \omega^{2} f$, $\epsilon=m_{e} / M, \mu=m m_{e} / M$ and $M=m+m_{e}$.

## 3. Metamorphoses of the amplitude profiles

To find approximate solutions we insert into Eq. (3) $y(t)=$ $A \cos (\omega t+\varphi)+\varepsilon y_{1}+\ldots$. After appropriate steps of the KBM procedure [2] we get implicit equation for amplitude profile, $F\left(\omega^{2}, A^{2} ; \Lambda\right)=0$, where $\Lambda$ denotes parameters.

Singular points of the amplitude curve $F(X, Y ; \Lambda)=0$, where $X=\omega^{2}, Y=A^{2}$, are computed from equations [4]:

$$
\begin{equation*}
F=0, \quad \frac{\partial F}{\partial X}=0, \quad \frac{\partial F}{\partial Y}=0 \tag{4}
\end{equation*}
$$

Singular points are interesting because amplitude profiles change their form in the vicinity of such points and so do the bifurcation diagrams. More complicated metamorphoses of the amplitude curves are possible near degenerate singular points for which the Hessian vanishes [4]:
$\frac{\partial^{2} F}{\partial X^{2}} \frac{\partial^{2} F}{\partial Y^{2}}-\left(\frac{\partial^{2} F}{\partial X \partial Y}\right)^{2}=0$.

## 4. Amplitude profiles and bifurcation diagrams

Applying KBM method to the equation (3) we obtain approximate formula $y(t)=A \cos (\omega t+\varphi)$ where dependence of $A$ on $\omega$ is given by implicit equation $F\left(\omega^{2}, A^{2} ; \Lambda\right)=0$. The form of the function $F$ can be found in Ref. [3], cf. Eq. (4.1). We have solved equations (4), (5) obtaining amplitude profile with a cusp - degenerate singular point. In the neighbourhood in the parameter space of this solution there are two kinds of singular points: isolated points and self-intersections which merge to yield the cusp. Therefore, in the neighbourhood of cusp there are two basic scenarios when parameters converge to the cusp: isolated

[^0]points converging to the cusp, and self-intersections converging to the same degenerate singular point. There are also many mixed scenarios - an isolated point may become a self-intersection, then can be transformed into an isolated point again, etc.


Figure 1: Three amplitude curves with isolated points, black, dark gray, and gray, converging to the cusp.


Figure 2: Bifurcation diagrams, the first scenario. The curves correspond to the profiles of the same color in Fig. 1.

Let us start with the first scenario (computational procedures are described in Ref. [3]). In Fig. 1 the amplitude profiles with isolated points converging to the degenerate point are shown. The amplitudes are shown just after the formation of isolated points, and thus there are small ovals rather than points in Fig. 1. In Fig. 2 the corresponding bifurcation diagrams are displayed. There are three small branches visible in the bottom of the diagram, corresponding to these three ovals.

In the second scenario the case of the amplitude profiles with self-intersections is considered. In Fig. 3 two amplitude profiles just before formation of self-intersections, converging to the degenerate point, are shown. The bifurcation diagrams are displayed in Fig. 4. Since the amplitude profiles shown in Fig. 3 are disconnected there are discontinuities of the branches in the bifurcation diagrams.


Figure 3: Two amplitude curves with self-intersections, black and gray, converging to the cusp.


Figure 4: Bifurcation diagrams, the second scenario. The curves correspond to the profiles of the same color in Fig. 3.

## 5. Closing remarks

We have demonstrated, that in the neighbourhood of a degenerate singular point (a cusp) a very complicated dynamics emerges with two basic scenarios of convergence to the cusp and many mixed scenarios possible.

## References

[1] Den Hartog, J. P., Mechanical Vibrations, 4th edition, Dover Publications, New York, 1985.
[2] Nayfeh, A. H., Introduction to Perturbation Techniques, John Wiley \& Sons, New York, 1981.
[3] Kyzioł, J., Okniński, A., Exact nonlinear fourth-order equation for two coupled oscillators: metamorphoses of resonance curves, Acta Phys. Polon. B 44, pp. 35-47, 2013.
[4] C. T. C. Wall, Singular Points of Plane Curves, Cambridge University Press, New York 2004.


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